

Differential Fay identities and auxiliary linear problem of integrable hierarchies

Kanehisa Takasaki *

Graduate School of Human and Environmental Studies
Kyoto University
Yoshida, Sakyo, Kyoto 606-8501, Japan
E-mail: takasaki@math.h.kyoto-u.ac.jp

Abstract

We review the notion of differential Fay identities and demonstrate, through case studies, its new role in integrable hierarchies of the KP type. These identities are known to be a convenient tool for deriving dispersionless Hirota equations. We show that differential (or, in the case of the Toda hierarchy, difference) Fay identities play a more fundamental role. Namely, they are nothing but a generating functional expression of the full set of auxiliary linear equations, hence substantially equivalent to the integrable hierarchies themselves. These results are illustrated for the KP, Toda, BKP and DKP hierarchies. As a byproduct, we point out some new features of the DKP hierarchy and its dispersionless limit.

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1 Introduction

The simplest, but most important example of dispersionless limit can be seen in the KdV equation

$$\partial_t u + 6u\partial_x u + \epsilon^2 \partial_3^3 u = 0$$

with a small dispersion parameter ϵ^2 in front of the dispersion term. By setting $\epsilon = 0$, the third order PDE reduces to the first order PDE (the dispersionless KdV equation)

$$\partial_t u + 6u\partial_x u = 0.$$

This naive procedure changes the properties of solutions drastically. For example, since soliton solutions can exist in the balance of nonlinearity and dispersion, the dispersionless KdV equation no longer has soliton-like solutions. Nonlinearity without dispersion leads to the so called gradient catastrophe, namely, a regular initial value develops singularity (“shock”) with $|\partial_x u| = \infty$. This means that the approximation by the dispersionless equation breaks down at that stage. In the presence of a small but nonzero dispersion term, this singularity is regularized to form an oscillatory region called “dispersive shock” (see the review by Lax, Levermore and Venakides [1]). Nevertheless the dispersionless KdV equation is a good approximation until the gradient catastrophe takes place. Moreover, this equation inherits many aspects of integrability of the KdV equation such as the existence of a Lax formalism, an infinite number of commuting flows, etc. We can consider a dispersionless limit of the KP equation

$$\partial_y^2 u = \partial_x (\partial_t u + u\partial_x u + \epsilon^2 \partial_3^3 u)$$

in a similar manner.

Another significant example is the two-dimensional Toda equation. This is originally a system of PDEs

$$\partial_x \partial_y \phi_s + e^{\phi_{s+1} - \phi_s} - e^{\phi_s - \phi_{s-1}} = 0$$

for an infinite number of fields $\phi_s = \phi_s(x, y)$ indexed by an integer $s \in \mathbf{Z}$. x and y are light cone coordinates of the two-dimensional Minkowski spacetime. If s is a coordinate on a one-dimensional lattice, and ϕ_s is interpreted as a field $\phi(s) = \phi(s, x, y)$ on a partially discretized three-dimensional spacetime, we can consider a continuum limit as the lattice spacing ϵ tends to 0. To this end, we restart from the rescaled Toda equation

$$\epsilon \partial_x \partial_y \phi(s) + e^{(\phi(s+\epsilon) - \phi(s))/\epsilon} - e^{(\phi(s) - \phi(s-\epsilon))/\epsilon} = 0$$

and let $\epsilon \rightarrow 0$. This yields the dispersionless Toda equation

$$\partial_x \partial_y \phi + \partial_s (e^{\partial_s \phi}) = 0.$$

Remarkably, this equation is also known in general relativity as the Boyer-Finley equation, which was discovered in a classification of selfdual spacetimes with a Killing symmetry [2, 3]. This equation, too, is a kind of integrable system.

These rather naive constructions of dispersionless limit can be reformulated in a more systematic way in the framework of the KdV, KP and Toda hierarchies. The outcome are the the dispersionless KdV, KP and Toda hierarchies. These dispersionless integrable hierarchies share many properties with the original dispersive systems. We refer details to the review by Takasaki and Takebe [4] and recall a few essential features.

Firstly, the dispersionless limit turns out to be an analogue of “quasi-classical limit” in quantum mechanics. The small parameter ϵ amounts to the Planck constant \hbar . This analogy (quantum-classical correspondence) becomes particularly fruitful when one considers the Lax formalism in the dispersionless limit. In the dispersionless KP and Toda hierarchies, commutators of (pseudo)differential operators in the Lax and Zakharov-Shabat equations are replaced by Poisson brackets, and the role of auxiliary linear equations are played by Hamilton-Jacobi equations. Actually, these ideas are rather old and can be found in studies of the Benney hierarchy in the early eighties [5, 6, 7].

Secondly, one can formulate the quasi-classical limit in terms of the tau function as well. This is based on ideas borrowed from random matrices, topological field theory and string theory [8, 9, 10]. Although the Hirota equations themselves do not survive the dispersionless limit, one can find an alternative framework, namely, “dispersionless Hirota equations” [11, 4, 12]. Actually, we can see a prototype of dispersionless Hirota equations in the aforementioned dispersionless KdV and Toda equations. In the case of the dispersionless KdV equation, we can convert the equation to

$$\partial_t \partial_x F + 3(\partial_x^2 F)^2 = 0$$

by changing variables as $u = 2\partial_x^2 F$ and integrating the equation once with respect to x . In the case of the dispersionless Toda equation, we can similarly derive the equation

$$\partial_x \partial_y F + e^{\partial_s^2 F} = 0$$

by substituting $\phi = \partial_s F$. The new dependent variable F may be thought of as a counterpart of the tau function in the dispersionless systems. Note that, unlike Hirota equations in the dispersive case, the equations for the F function are no longer bilinear.

Dispersionless Hirota equations have played a central role in recent studies on dispersionless integrable hierarchies. They were applied to various problems of mathematical physics such as interface dynamics [13], associativity equations [14] and string field theory [15, 16, 17, 18]. Alongside these applications, mathematical aspects of dispersionless Hirota equations were also studied in detail [19, 20, 21, 22, 23, 24, 25, 26]. It has been established through these studies that dispersionless Hirota equations can be a fundamental language for dispersionless integrable hierarchies.

By the way, the notion of dispersionless Hirota equations was first discovered (in the case of the KP hierarchy [4, Appendix B]) as dispersionless limit of the so called “differential Fay identity”. In other words, this identity (introduced by Adler and van Moerbeke [27] in a quite different context) is a dispersive counterpart of dispersionless Hirota equations. This raises a natural question: Can the notion of differential Fay identity play the same fundamental role as that of dispersionless Hirota equations?

This paper presents several case studies on this question. In the case of the KP hierarchy, this question was already answered affirmatively when the notion of dispersionless Hirota equations was first proposed [4]. It was proven therein that the differential Fay identity, which is a consequence of the KP hierarchy, is actually equivalent to the KP hierarchy itself. We shall review this result, and point out that the most essential part of this result is the fact that the differential Fay identity is a generating functional expression of an infinite number of auxiliary linear equations. We shall show that the same point of view is valid for some other cases as well.

This paper is organized as follows. Sections 2 and 3 are devoted to the most fundamental cases, namely, the KP and Toda hierarchies. All results presented here are well known and can be found in the literature. The case of the KP hierarchy is reviewed in detail (Section 2) as a prototype of the subsequent cases. In the case of the Toda hierarchy (Section 3), difference analogues of differential Fay identities show up. Sections 4 and 5 deal with the BKP and DKP hierarchies, which are relatives of the KP hierarchy. Most results on the BKP hierarchy (Section 4) are already published except for the relation between the differential Fay identities and the auxiliary linear problem. The results on the DKP hierarchy (Section 5) are mostly new. In Section 6 we show our conclusion along with several remarks.

2 KP hierarchy and differential Fay identity

2.1 Bilinear equations for tau function

Let $\mathbf{t} = (t_1, t_2, \dots)$ denote the set of time variables of the KP hierarchy. The first one t_1 is identified with the spatial variable x in the Lax formalism.

The tau function $\tau = \tau(\mathbf{t})$ of the KP hierarchy [28, 29, 30, 31] satisfies an infinite number of bilinear equations. These equations can be written in a compact form as

$$(D_1^4 - 4D_1D_3 + 3D_2^2)\tau \cdot \tau = 0, \quad \dots \quad (2.1)$$

with Hirota’s notation

$$P(D_1, D_2, \dots)\tau \cdot \tau = P(\partial'_1 - \partial_1, \partial'_2 - \partial_2, \dots)\tau(t'_1, t'_2, \dots)\tau(t_1, t_2, \dots)|_{\mathbf{t}'=\mathbf{t}},$$

where ∂_n and ∂'_n denote the derivatives $\partial_n = \partial/\partial t_n$, $\partial'_n = \partial/\partial t'_n$ in the component of \mathbf{t} and $\mathbf{t}' = (t'_1, t'_2, \dots)$.

Date, Jimbo, Kashiwara and Miwa discovered that these Hirota equations can be encoded to a single equation of the form

$$\oint \frac{dz}{2\pi i} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(\mathbf{t}' - [z^{-1}]) \tau(\mathbf{t} + [z^{-1}]) = 0, \quad (2.2)$$

where $\xi(\mathbf{t}, z)$ and $[\alpha]$ are the standard notions

$$\xi(\mathbf{t}, z) = \sum_{n=1}^{\infty} t_n z^n, \quad [\alpha] = \left(\alpha, \frac{\alpha^2}{2}, \dots, \frac{\alpha^n}{n}, \dots \right),$$

and \oint is the contour integral along a sufficiently large circle $|z| = R$ (or a formal algebraic operator extracting the coefficient of z^{-1} of a Laurent series). (2.2) is understood to hold for arbitrary values of \mathbf{t}' and \mathbf{t} . More precisely, (2.2) is a generating functional expression of an infinite number of equations that can be obtained by Taylor expansion at $\mathbf{t}' = \mathbf{t}$.

One can convert (2.2) into a form that is more directly related to Hirota equations. This is achieved by introducing a new set of variables $\mathbf{a} = (a_1, a_2, \dots)$ and substituting

$$\mathbf{t}' \rightarrow \mathbf{t} - \mathbf{a}, \quad \mathbf{t} \rightarrow \mathbf{t} + \mathbf{a}.$$

(2.2) thereby takes the form

$$\oint \frac{dz}{2\pi i} e^{-2\xi(\mathbf{a}, z)} \tau(\mathbf{t} + \mathbf{a} + [z^{-1}]) \tau(\mathbf{t} - \mathbf{a} - [z^{-1}]) = 0.$$

It is convenient to introduce the elementary Schur functions $h_n(\mathbf{t})$, $n = 0, 1, \dots$ here. They are defined by an exponential generating function as

$$e^{\xi(\mathbf{t}, z)} = \sum_{n=0}^{\infty} h_n(\mathbf{t}) z^n.$$

One can thereby express the exponential factor $e^{-2\xi(\mathbf{a}, z)}$ as

$$e^{-2\xi(\mathbf{a}, z)} = \sum_{n=0}^{\infty} h_n(-2\mathbf{a}) z^n.$$

Moreover, the product of two shifted tau functions can be expressed as

$$\begin{aligned} \tau(\mathbf{t} + \mathbf{a} + [z^{-1}]) \tau(\mathbf{t} - \mathbf{a} - [z^{-1}]) &= \exp \left(\sum_{n=1}^{\infty} \left(a_n + \frac{z^{-n}}{n} \right) D_n \right) \tau(\mathbf{t}) \cdot \tau(\mathbf{t}) \\ &= \sum_{n=0}^{\infty} h_n(\tilde{D}_{\mathbf{t}}) z^{-n} e^{\langle \mathbf{a}, D_{\mathbf{t}} \rangle} \tau(\mathbf{t}) \cdot \tau(\mathbf{t}), \end{aligned}$$

where

$$\tilde{D}_{\mathbf{t}} = \left(D_1, \frac{D_2}{2}, \dots, \frac{D_n}{n}, \dots \right), \quad \langle \mathbf{a}, D_{\mathbf{t}} \rangle = \sum_{n=1}^{\infty} a_n D_n.$$

(2.2) can be thus eventually converted to the Hirota form

$$\sum_{n=0}^{\infty} h_n(-2\mathbf{a})h_{n+1}(\tilde{D}\mathbf{t})e^{\langle \mathbf{a}, D\mathbf{t} \rangle} \tau(\mathbf{t}) \cdot \tau(\mathbf{t}) = 0. \quad (2.3)$$

Note that this procedure is reversible; one can trace it back and recover (2.2) from (2.3).

The last equation (2.3) is a generating functional form of an infinite number of Hirota equations, which are obtained by expanding (2.3) in powers of \mathbf{a} . For example, the terms linear in a_n 's gives the special Hirota equations

$$\left(D_1 D_n - 2h_{n+1}(\tilde{D}\mathbf{t}) \dots \right) \tau(\mathbf{t}) \cdot \tau(\mathbf{t}) = 0, \quad n = 3, 4, \dots \quad (2.4)$$

Note that those for $n = 1$ and $n = 2$ are trivial identities. The equation for $n = 3$ is exactly the lowest Hirota equation in (2.1).

2.2 Fay-type identities

The following identity, referred to as the Fay identity in the following, holds for all tau functions of the KP hierarchy:

$$\begin{aligned} & (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \tau(\mathbf{t} + [\lambda_1^{-1}] + [\lambda_2^{-1}]) \tau(\mathbf{t} + [\lambda_3^{-1}] + [\lambda_4^{-1}]) \\ & - (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) \tau(\mathbf{t} + [\lambda_1^{-1}] + [\lambda_3^{-1}]) \tau(\mathbf{t} + [\lambda_2^{-1}] + [\lambda_4^{-1}]) \\ & + (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3) \tau(\mathbf{t} + [\lambda_1^{-1}] + [\lambda_4^{-1}]) \tau(\mathbf{t} + [\lambda_2^{-1}] + [\lambda_3^{-1}]) = 0. \end{aligned} \quad (2.5)$$

$\lambda_1, \lambda_2, \lambda_3$ and λ_4 are arbitrary parameters. This is a generalization of Fay's trisecant identities for the Riemann theta function and the prime form on an arbitrary Riemann surface. Sato and Sato [29] derived this identity and its generalizations as a consequence of Plücker relations on an underlying Grassmann manifold. One can derive it from the bilinear identity (2.2) as well, though we omit details. We shall show a similar procedure for the differential Fay identity below.

Adler and van Moerbeke [27] obtained the differential Fay identity

$$\begin{aligned} & (\lambda - \mu) \tau(\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]) \tau(\mathbf{t}) - (\lambda - \mu) \tau(\mathbf{t} + [\lambda^{-1}]) \tau(\mathbf{t} + [\mu^{-1}]) \\ & + (\partial_1 \tau)(\mathbf{t} + [\lambda^{-1}]) \tau(\mathbf{t} + [\mu^{-1}]) - (\partial_1 \tau)(\mathbf{t} + [\mu^{-1}]) \tau(\mathbf{t} + [\lambda^{-1}]) = 0 \end{aligned} \quad (2.6)$$

by a confluence procedure letting two of λ 's in the Fay identity tend to 0. We now derive this identity from the bilinear identity (2.2). Firstly, let us differentiate the bilinear equation by t'_1 . This gives the equation

$$\oint \frac{dz}{2\pi i} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \left(z \tau(\mathbf{t}' - [z^{-1}]) + (\partial_1 \tau)(\mathbf{t}' - [z^{-1}]) \right) \tau(\mathbf{t} + [z^{-1}]) = 0.$$

Secondly, assuming that λ and μ sit on the far side of the contour $|z| = R$ of the integral (i.e., $|\lambda| > R$ and $|\mu| > R$), we specialize \mathbf{t}' as

$$\mathbf{t}' = \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}].$$

Then, by virtue of the identity

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z}{w} \right)^n = -\log \left(1 - \frac{z}{w} \right),$$

the exponential factor $e^{\xi(\mathbf{t}' - \mathbf{t}, z)}$ reduces to a rational function of z, λ, μ as

$$e^{\xi(\mathbf{t}' - \mathbf{t}, z)} = \left(1 - \frac{z}{\lambda} \right)^{-1} \left(1 - \frac{z}{\mu} \right)^{-1} = \frac{\lambda \mu}{(z - \lambda)(z - \mu)}.$$

The foregoing equation thereby becomes

$$\oint \frac{dz}{2\pi i} \frac{\lambda \mu}{(z - \lambda)(z - \mu)} \left(z \tau(\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \right. \\ \left. + (\partial_1 \tau)(\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \tau(\mathbf{t} + [z^{-1}]) \right) = 0.$$

Now assuming that $\tau(\mathbf{t})$ is a holomorphic function of \mathbf{t} in a neighborhood of $\mathbf{t} = \mathbf{0}$, we conclude that the integrand is a meromorphic function of z in $|z| \geq R$ with poles at $z = \infty, \lambda, \mu$. Therefore we can use residue calculus to calculate the contour integral as a sum of the residues at the poles. This yields (2.6).

(2.6) is not very convenient for some purposes. It is more suggestive to rewrite it as

$$\frac{\tau(\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]) \tau(\mathbf{t})}{\tau(\mathbf{t} + [\lambda^{-1}]) \tau(\mathbf{t} + [\mu^{-1}])} = 1 - \frac{1}{\lambda - \mu} \partial_1 \log \frac{\tau(\mathbf{t} + [\lambda^{-1}])}{\tau(\mathbf{t} + [\mu^{-1}])}. \quad (2.7)$$

Moreover, one can use the differential operator

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_n$$

to rewrite it further as

$$\exp \left((e^{D(\lambda)} - 1)(e^{D(\mu)} - 1) \log \tau \right) = 1 - \frac{\partial_1 (e^{D(\lambda)} - e^{D(\mu)}) \log \tau}{\lambda - \mu}. \quad (2.8)$$

As proven by Takasaki and Takebe [4], the differential Fay identity (2.6) is equivalent to the KP hierarchy itself. A clue of the proof is a close relationship between the differential Fay identity and the auxiliary linear problem.

2.3 Auxiliary linear equations for wave functions

The auxiliary linear problem for the KP hierarchy [28, 29, 30, 31] consists of the linear equations

$$(\partial_n - B_n) \Psi = 0, \quad (2.9)$$

where B_n 's are differential operators of the form

$$B_n = \partial_1^n + b_{n,2}\partial_1^{n-2} + \cdots + b_{n,n}.$$

These operators (the Zakharov-Shabat operators) satisfy the Zakharov-Shabat equations

$$[\partial_m - B_m, \partial_n - B_n] = 0, \quad (2.10)$$

equivalently,

$$\partial_n(B_m) - \partial_m(B_n) + [B_m, B_n] = 0,$$

where $\partial_n(B_m)$ means differentiating the coefficients of B_m by t_n . Moreover, there is a pseudo-differential operator (Lax operator) of the form

$$L = \partial_1 + \sum_{j=1}^{\infty} u_{j+1}\partial_1^{-j}$$

such that the Zakharov-Shabat operators are expressed as

$$B_n = (L^n)_{\geq 0}, \quad (2.11)$$

where $(\)_{\geq 0}$ stands for the part of nonnegative powers of ∂_1 . The Lax operator satisfies the Lax equations

$$\partial_n(L) = [B_n, L]. \quad (2.12)$$

In analogy with quantum mechanics, solutions of the auxiliary linear equations are called “wave functions”. The most fundamental wave function can be obtained from the tau function by the formula

$$\Psi(\mathbf{t}, z) = \frac{\tau(\mathbf{t} - [z^{-1}])}{\tau(\mathbf{t})} e^{\xi(\mathbf{t}, z)} = \frac{e^{-D(z)}\tau(\mathbf{t})}{\tau(\mathbf{t})} e^{\xi(\mathbf{t}, z)}.$$

As an immediate consequence of the bilinear equation (2.2) for the tau function, this wave function and its dual

$$\Psi^*(\mathbf{t}, z) = \frac{\tau(\mathbf{t} + [z^{-1}])}{\tau(\mathbf{t})} e^{-\xi(\mathbf{t}, z)} = \frac{e^{D(z)}\tau(\mathbf{t})}{\tau(\mathbf{t})} e^{-\xi(\mathbf{t}, z)}$$

satisfy the bilinear equation

$$\oint \frac{dz}{2\pi i} \Psi(\mathbf{t}', z) \Psi^*(\mathbf{t}, z) = 0. \quad (2.13)$$

We can derive the auxiliary linear equations (2.9) from this bilinear equation as follows [33, 34]. Differentiating (2.13) by t'_n yield the new bilinear equation

$$\oint \frac{dz}{2\pi i} (\partial_n \Psi)(\mathbf{t}', z) \Psi^*(\mathbf{t}, z) = 0.$$

In a similar manner, bilinear equations of the form

$$\oint \frac{dz}{2\pi i} (B\Psi)(\mathbf{t}', z) \Psi^*(\mathbf{t}, z) = 0$$

hold for any higher order differential operator $B = B(\partial_1)$ with respect to t_1 . (Actually, the bilinear equation (2.13) should be interpreted as a generating functional form of all those equations.) Consequently, we obtain the equation

$$\oint \frac{dz}{2\pi i} R(\mathbf{t}, z) \Psi^*(\mathbf{t}, z) = 0 \quad (2.14)$$

for the linear combination

$$R(\mathbf{t}, z) = \partial_n \Psi(\mathbf{t}, z) - B\Psi(\mathbf{t}, z).$$

We now choose B to be a differential operator $B_n = B_n(\partial_1)$ of the aforementioned form such that

$$R(\mathbf{t}, z) = \partial_n \Psi(\mathbf{t}, z) - B_n(\partial_1) \Psi(\mathbf{t}, z) = O(z^{-1}) e^{\xi(\mathbf{t}, z)}, \quad (2.15)$$

namely, the Laurent expansion of $R(\mathbf{t}, z) e^{-\xi(\mathbf{t}, z)}$ at $z = \infty$ is of order $O(z^{-1})$. Such an operator $B_n(\partial_1)$ does exist and is unique (see below for an explicit construction of this operator). On the other hand, there is a general formula that relates a contour integral of the form

$$\oint \frac{dz}{2\pi i} (P(x', \partial'_x) e^{x'z}) (Q(x, \partial_x) e^{-xz})$$

to the product of $P(x, \partial_x)$ and $Q(x, \partial_x)^*$ (the formal adjoint of $Q(x, \partial_x)$) [33, 34]. With the aid of that formula, we can prove that any function $R(\mathbf{t}, z)$ with these properties (2.14) and (2.15) has to vanish identically. Thus $\Psi(\mathbf{t}, z)$ turns out to satisfy the auxiliary linear equations (2.9).

Having obtained the auxiliary linear equations for $\Psi(\mathbf{t}, z)$, we can deduce the existence of the Lax operator as follows. A clue is the so called dressing (or Sato-Wilson) operator

$$W = 1 + \sum_{j=1}^{\infty} w_j \partial_1^{-j}$$

defined by the coefficients of the Laurent expansion

$$\Psi(\mathbf{t}, z) e^{-\xi(\mathbf{t}, z)} = 1 + \sum_{j=1}^{\infty} w_j z^{-j}$$

of $\Psi(\mathbf{t}, z) e^{-\xi(\mathbf{t}, z)}$ at $z = \infty$. Note that $\Psi(\mathbf{t}, z)$, in turn, can be expressed as

$$\Psi(\mathbf{t}, z) = W e^{\xi(\mathbf{t}, z)}. \quad (2.16)$$

One can immediately see that the operators

$$B_n = (W\partial_1^n W^{-1})_{\geq 0}$$

turn out to fulfill the condition (2.15). Therefore if we define L as

$$L = W\partial_1 W^{-1},$$

the foregoing expression (2.11) of B_n 's is achieved. Moreover, by plugging (2.16) into the auxiliary linear equations, one finds that W satisfies the the Sato equations

$$\partial_n(W) = B_n W - W\partial_1^n = -(W\partial_1^n W^{-1})_{<0} W, \quad (2.17)$$

where $(\)_{<0}$ stands for the part of negative powers of ∂_1 . The Lax equations (2.12) readily follow from the Sato equations.

Note that the identification of the Lax and Zakharov-Shabat operators is independent of the foregoing derivation of the auxiliary linear equations. Namely, once one can anyhow show that $\Psi(\mathbf{t}, z)$ satisfies the auxiliary linear equations (2.9), one can thereby conclude the existence of an operator L which satisfies the Lax equations (2.12) and by which B_n 's are expressed as (2.11). This remark becomes technically important when we derive the auxiliary linear equations from the differential Fay identity.

2.4 Differential Fay identity and auxiliary linear equations

The foregoing derivation of the auxiliary linear equations (2.9) from the bilinear equation (2.13) is somewhat indirect. We now show that the differential Fay identity gives a more direct approach to the auxiliary linear equations.

Let us anyway rewrite the differential Fay identity in the language of the wave function. The first step is to shift \mathbf{t} in (2.7) as $\mathbf{t} \rightarrow \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}]$. (2.7) thereby changes as

$$\frac{\tau(\mathbf{t} - [\lambda^{-1}] - [\mu^{-1}])\tau(\mathbf{t})}{\tau(\mathbf{t} - [\lambda^{-1}])\tau(\mathbf{t} - [\mu^{-1}])} = 1 + \frac{1}{\lambda - \mu} \partial_1 \log \frac{\tau(\mathbf{t} - [\lambda^{-1}])}{\tau(\mathbf{t} - [\mu^{-1}])}.$$

The next step is to multiply both hand sides of the equation by $(\lambda - \mu)e^{\xi(\mathbf{t}, \mu)}\tau(\mathbf{t} - [\mu^{-1}])/\tau(\mathbf{t})$. This yields the equation

$$\begin{aligned} & (\lambda - \mu)e^{\xi(\mathbf{t}, \mu)}e^{-D(\lambda)}\frac{\tau(\mathbf{t} - [\mu^{-1}])}{\tau(\mathbf{t})} \\ &= \left(\lambda + \partial_1 \log \frac{\tau(\mathbf{t} - [\lambda^{-1}])}{\tau(\mathbf{t})} - \mu - \partial_1 \log \frac{\tau(\mathbf{t} - [\mu^{-1}])}{\tau(\mathbf{t})} \right) \times \\ & \quad \times e^{\xi(\mathbf{t}, \mu)}\frac{\tau(\mathbf{t} - [\mu^{-1}])}{\tau(\mathbf{t})}. \end{aligned}$$

Noting that the commutator of $e^{-D(\lambda)}$ and $e^{\xi(\mathbf{t}, \mu)}$ yields the factor $1 - \mu/\lambda$, we can rewrite the last equation as

$$\lambda e^{-D(\lambda)} \Psi(\mathbf{t}, \mu) = (\partial_1 \log \Psi(\mathbf{t}, \lambda) - \partial_1) \Psi(\mathbf{t}, \mu). \quad (2.18)$$

Expanded in powers of λ , this equation yields an infinite number of linear equations of the form

$$h_n(-\tilde{\partial}_{\mathbf{t}}) \Psi(\mathbf{t}, \mu) = v_n \Psi(\mathbf{t}, \mu) \quad (2.19)$$

for $n = 2, 3, \dots$, where $\tilde{\partial}_{\mathbf{t}}$ is defined as

$$\tilde{\partial}_{\mathbf{t}} = \left(\partial_1, \frac{\partial_2}{2}, \dots, \frac{\partial_n}{n}, \dots \right),$$

and v_n 's are the coefficients of the Laurent expansion of $\partial_1 \log \Psi(\mathbf{t}, \lambda)$, namely,

$$\partial_1 \log \Psi(\mathbf{t}, \lambda) = \lambda + \sum_{n=1}^{\infty} v_{n+1} \lambda^{-n}.$$

Linear equations (2.19) were first discovered by Sato and Sato [29], but implications of these equations remained to be clarified for years. It was then pointed out by Takasaki and Takebe [4] that (2.19) are equivalent to the standard auxiliary linear problem (2.9). As one can immediately see from the first few equations

$$\begin{aligned} \left(-\frac{1}{2} \partial_2 + \frac{1}{2} \partial_1^2 \right) \Psi(\mathbf{t}, \mu) &= v_2 \Psi(\mathbf{t}, \mu), \\ \left(-\frac{1}{3} \partial_3 + \frac{1}{2} \partial_2 \partial_1 - \frac{1}{6} \partial_1^3 \right) \Psi(\mathbf{t}, \mu) &= v_3 \Psi(\mathbf{t}, \mu), \quad \dots, \end{aligned}$$

the left hand side of the n -th equation of (2.19) takes such a form as

$$h_n(-\tilde{\partial}_{\mathbf{t}}) \Psi(\mathbf{t}, \mu) = \left(-\frac{1}{n} \partial_n + C_n(\partial_1, \dots, \partial_{n-1}) \right) \Psi(\mathbf{t}, \mu),$$

where $C_n(\partial_1, \dots, \partial_{n-1})$ is a differential operator that contains $\partial_1, \dots, \partial_{n-1}$. One can recursively eliminate the operators $\partial_2, \dots, \partial_{n-1}$ other than ∂_1 using the preceding equations $h_m(-\tilde{\partial}_{\mathbf{t}}) \Psi(\mathbf{t}, z) = v_m \Psi(\mathbf{t}, z)$ for $m < n$. The outcome are linear equations of the usual evolutionary form

$$\begin{aligned} \partial_2 \Psi(\mathbf{t}, \mu) &= (\partial_1^2 - 2v_2) \Psi(\mathbf{t}, \mu), \\ \partial_3 \Psi(\mathbf{t}, \mu) &= (\partial_1^3 - 3v_2 \partial_1 - 2\partial_1 v_2 - 3v_3) \Psi(\mathbf{t}, \mu), \quad \dots \end{aligned}$$

One can identify the operators on the right hand side to be the standard Zakharov-Shabat operators (2.11) by the same reasoning as we have used for deriving the auxiliary linear equations (2.9) from the bilinear equation (2.13). Thus (2.19) turns out to be equivalent to (2.9). In other words, (2.18) is a generating functional form of these linear equations.

We are thus led to the conclusion that the differential Fay identity is actually the auxiliary linear problem in disguise. This fact lies in the heart of the proof [4] of the fact that the differential Fay identity is equivalent to the full system of the KP hierarchy.

Let us note that v_n 's are rather well known quantities, namely, conserved densities of the KP hierarchy [35]. Since $L\Psi(\mathbf{t}, \lambda) = \lambda\Psi(\mathbf{t}, \lambda)$, one can rewrite the foregoing defining equation of v_n 's as

$$\begin{aligned}\partial_1\Psi(\mathbf{t}, \lambda) &= \left(\lambda + \sum_{n=1}^{\infty} v_{n+1}\lambda^{-n}\right)\Psi(\mathbf{t}, \lambda) \\ &= \left(L + \sum_{n=1}^{\infty} v_{n+1}L^{-n}\right)\Psi(\mathbf{t}, \lambda).\end{aligned}$$

This implies the operator identity

$$\partial_1 = L + \sum_{n=1}^{\infty} v_{n+1}L^{-n}. \quad (2.20)$$

Moreover, since

$$\log \Psi(\mathbf{t}, \lambda) = \xi(\mathbf{t}, \lambda) + (e^{-D(\lambda)} - 1) \log \tau,$$

one has the explicit formula

$$v_n = \partial_1 h_n(-\tilde{\partial}_{\mathbf{t}}) \log \tau. \quad (2.21)$$

The differential Fay identity is particularly useful for studying the limit to the dispersionless KP hierarchy. This limit is achieved by a kind of “quasi-classical” limit. One can formulate this procedure in both the Hirota formalism and the Lax formalism. Let us now turn to this subject.

2.5 Quasi-classical limit in Hirota formalism

In the Hirota formalism, the quasi-classical limit is formulated as follows [36]. Firstly, we allow the tau function to depend on an extra small parameter \hbar (an analogue of the Planck constant in quantum mechanics) as $\tau = \tau(\hbar, \mathbf{t})$. Secondly, we assume that the rescaled tau function $\tau_{\hbar}(\mathbf{t}) = \tau(\hbar, \hbar^{-1}\mathbf{t})$ behaves as

$$\tau_{\hbar}(\mathbf{t}) = \exp\left(\hbar^{-2}F(\mathbf{t}) + O(\hbar^{-1})\right) \quad (2.22)$$

in the limit of $\hbar \rightarrow 0$. $F(\mathbf{t})$ is a key function in the theory of dispersionless integrable systems, having several different names such as “dispersionless tau function”, “free energy”, “prepotential”, etc. that stem from its origin in problems of mathematical physics [8, 9, 10].

By rescaling the tau function as above, the differential Fay identity (2.8) takes the rescaled form

$$\exp\left((e^{\hbar D(\lambda)} - 1)(e^{\hbar D(\mu)} - 1) \log \tau_{\hbar}\right) = 1 - \hbar \frac{\partial_1(e^{\hbar D(\lambda)} - e^{\hbar D(\mu)}) \log \tau_{\hbar}}{\lambda - \mu}.$$

Under the foregoing ansatz (2.22), both hand sides of this equation has a finite limit as $\hbar \rightarrow 0$. This yields the equation [4]

$$e^{D(\lambda)D(\mu)F} = 1 - \frac{\partial_1(D(\lambda) - D(\mu))F}{\lambda - \mu} \quad (2.23)$$

for the F function $F = F(\mathbf{t})$.

The last equation, known as the dispersionless Hirota equation, is a generating functional form of an infinite number of equations, which are obtained by Laurent expansion of both hand sides in powers of λ and μ . It should be noted that “the dispersionless Hirota equation” might be a misleading name, because (2.23) is neither a direct limit of the Hirota equations (2.1) themselves nor of their generating functional form (2.2), but of the differential Fay identity. Therefore, it would be better to call (2.23) “the dispersionless differential Fay identity”.

In this respect, Carroll and Kodama [12] observed an interesting fact: They pointed out that the special Hirota equations (2.4) have a direct quasi-classical limit. To see this fact, it is convenient to start from the generating functional form

$$2e^{D(\mu)}\tau \cdot \tau = 2 + \sum_{n=1}^{\infty} \mu^{-n-1} D_1 D_n \tau \cdot \tau$$

of (2.4). It is not hard to see that this equation survives the quasi-classical limit and yields the equation

$$e^{D(\mu)^2 F} = 1 + \sum_{n=1}^{\infty} \mu^{-n-1} \partial_1 \partial_n F \quad (2.24)$$

for the F function. Actually, the last equation can also be obtained by letting $\lambda \rightarrow \mu$ in (2.23). Expanded in powers of μ , this equation generates an infinite number of equations of the form

$$\partial_1 \partial_n F = h_{n+1}(Z_1, Z_2, \dots), \quad n = 1, 2, \dots, \quad (2.25)$$

where

$$Z_n = \sum_{j+k=n} \frac{\partial_j \partial_k F}{jk}.$$

Carroll and Kodama studied the algebraic structure of these genuine “dispersionless Hirota equations” in detail.

As the differential Fay identity is equivalent to the KP hierarchy itself, its dispersionless analogue (2.23) turns out to be equivalent to (the Lax formalism of) the dispersionless KP hierarchy [14, 24, 25]. This is by no means an obvious fact. We shall return to this issue later on.

2.6 Quasi-classical limit in Lax formalism

In the Lax formalism, the procedure of quasi-classical limit starts from the WKB ansatz [37, 38]

$$\Psi_{\hbar}(\mathbf{t}, z) = \exp\left(\hbar^{-1}S(\mathbf{t}, z) + O(\hbar^0)\right) \quad (2.26)$$

of the rescaled wave function $\Psi_{\hbar}(\mathbf{t}, z) = \Psi(\hbar, \hbar^{-1}\mathbf{t}, z)$. It is easy to see that this WKB ansatz follows from the quasi-classical ansatz (2.22) of the tau function. The associated S function is given by

$$S(\mathbf{t}, z) = \xi(\mathbf{t}, z) - D(z)F = \sum_{n=1}^{\infty} t_n z^n - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_n F.$$

The auxiliary linear equations for the rescaled wave function takes the rescaled form

$$\hbar \partial_n \Psi_{\hbar}(\mathbf{t}, z) = B_{\hbar, n}(\hbar \partial_1) \Psi_{\hbar}(\mathbf{t}, z),$$

where the coefficients of the operators

$$B_{\hbar, n}(\hbar \partial_1) = (\hbar \partial_1)^n + b_{\hbar, n, 2}(\hbar \partial_1)^{n-2} + \cdots + b_{\hbar, n, n}$$

have a smooth limit as $\hbar \rightarrow 0$. These equations may be thought of as analogues of the time-dependent Schrödinger equations in quantum mechanics. Consequently, the S function satisfies the Hamilton-Jacobi equations

$$\partial_n S(\mathbf{t}, z) = \mathcal{B}_n(\partial_1 S(\mathbf{t}, z)) \quad (2.27)$$

with the Hamiltonians

$$\mathcal{B}_n(p) = \lim_{\hbar \rightarrow 0} B_{\hbar, n}(p).$$

These stuff resemble the setup of quasi-classical approximation in quantum mechanics. The Zakharov-Shabat operators $B_{\hbar, n} = B_{\hbar, n}(\hbar \partial_1)$ are now replaced by functions $\mathcal{B}_n = \mathcal{B}_n(p)$ of a new variable p , which may be interpreted as the conjugate momentum of the coordinate $x = t_1$. By this “quantum-classical correspondence”, commutators of differential operators in the Lax and Zakharov-Shabat equations of the KP hierarchy are replaced by Poisson brackets

$$\{F, G\} = (\partial_p F)(\partial_1 G) - (\partial_1 F)(\partial_p G)$$

of functions on a two-dimensional phase space. We are thus led to the Zakharov-Shabat equations

$$\partial_n(\mathcal{B}_m) - \partial_m(\mathcal{B}_n) + \{\mathcal{B}_m, \mathcal{B}_n\} = 0 \quad (2.28)$$

and the Lax equations

$$\partial_n(\mathcal{L}) = \{\mathcal{B}_n, \mathcal{L}\} \quad (2.29)$$

with respect to Poisson brackets [37, 39, 40]. The Lax operator is also replaced by a function of the form

$$\mathcal{L} = p + \sum_{j=1}^{\infty} u_{j+1} p^{-j}.$$

\mathcal{B}_n 's are thereby expressed as

$$\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0},$$

where $(\)_{\geq 0}$ denotes the polynomial part of Laurent series of p .

The Hamilton-Jacobi equations (2.27) and the Zakharov-Shabat-Lax equations (2.28) – (2.29) can be transferred from one side to the other by the functional relation

$$p = \partial_1 S(\mathbf{t}, z) = z - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_1 \partial_n F \quad (2.30)$$

connecting the spectral parameter z and the momentum p . The inverse $p \mapsto z$ of the map $z \mapsto p$ is exactly the Lax function $\mathcal{L} = \mathcal{L}(p)$. One can see this fact from (2.21) and (2.20) as follows. By rescaling the tau function, (2.21) becomes

$$v_{\hbar, n} = \hbar \partial_1 h_n (-\hbar \tilde{\partial}_{\mathbf{t}}) \log \tau_{\hbar},$$

hence

$$\lim_{\hbar \rightarrow 0} v_{\hbar, n} = -\frac{1}{n} \partial_1 \partial_n F.$$

Consequently, (2.20) yields

$$p = \mathcal{L}(p) - \sum_{n=1}^{\infty} \frac{\mathcal{L}(p)^{-n}}{n} \partial_1 \partial_n F$$

in the limit as $\hbar \rightarrow 0$. Comparing this relation with (2.30), one finds that the inverse of the map $z \mapsto p = \partial_1 S(\mathbf{t}, z)$ is given by $z = \mathcal{L}(p)$.

2.7 Dispersionless Hirota equation and Hamilton-Jacobi equations

As already mentioned, the dispersionless Hirota equation (2.23) is equivalent to the Lax formalism of the dispersionless KP hierarchy [14, 24, 25]. Main part of the proof of this fact is to show that (2.23) is a generating functional form of the Hamilton-Jacobi equations (2.27). This is a dispersionless analogue of the relation between the differential Fay identity and the auxiliary linear equations.

The notion of “Faber polynomials” in complex analysis [25] plays a central role here. Let us demonstrate its power by assuming (2.23) and deriving (2.27) therefrom. We define $p(z)$ as

$$p(z) = \partial_1 S(\mathbf{t}, z)$$

and rewrite (2.23) as

$$e^{D(\lambda)D(\mu)F} = \frac{p(\lambda) - p(\mu)}{\lambda - \mu}.$$

Taking the logarithm of both hand sides yields

$$D(\lambda)D(\mu)F = \log \frac{p(\lambda) - p(\mu)}{\lambda - \mu}.$$

By virtue of the identity

$$\log\left(1 - \frac{\mu}{\lambda}\right) = - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\mu}{\lambda}\right)^n,$$

we can further rewrite the last equation as

$$\begin{aligned} \log \frac{p(\lambda) - p(\mu)}{\lambda} &= - \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \left(\mu^n - \sum_{m=1}^{\infty} \frac{\mu^{-m}}{m} \partial_n \partial_m F \right) \\ &= - \sum_{n=1}^{\infty} \frac{\lambda^{-n}}{n} \partial_n S(\mu). \end{aligned} \quad (2.31)$$

The proble is to solve this equation for $\partial_n S(\mu)$. To this end, we introduce the Farber polynomials $\Phi_n(p)$, $n = 1, 2, \dots$, of the map $z \mapsto p(z)$, which are uniquely defined by the generating functional relation

$$\log \frac{p(z) - q}{z} = - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Phi_n(q).$$

More explicitly, as we shall show below, $\Phi_n(p)$ are given by the polynomial part of the inverse map $p \mapsto z = z(p)$ as

$$\Phi_n(p) = \left(z(p)^n \right)_{\geq 0}. \quad (2.32)$$

In the present setting, $z(p)$ is nothing but $\mathcal{L}(p)$, hence

$$\Phi_n(p) = \mathcal{B}_n(p).$$

We can now solve (2.31) for $\partial_n S(\mu)$ as

$$\partial_n S(\mu) = \Phi_n(p(\mu)) = \mathcal{B}_n(p(\mu)). \quad (2.33)$$

The last equations are exactly the Hamilton-Jacobi equations (2.27).

Let us show that (2.32) indeed holds. To this end, we differentiate the defining equation of $\Phi_n(q)$ by z . This yields the identity

$$\frac{p'(z)}{p(z) - q} = z^{-1} + \sum_{n=1}^{\infty} z^{-n-1} \Phi_n(q),$$

where $p'(z)$ denotes the z -derivative $\partial_z p(z)$. We can extract $\Phi_n(q)$ by a contour integral as

$$\Phi_n(q) = \oint \frac{dz}{2\pi i} \frac{z^n p'(z)}{p(z) - q},$$

where the contour of integral is a sufficiently large circle $|z| = R$ such that $|q| < R$. Since the map $z \mapsto p(z)$ is invertible in a neighborhood of $z = \infty$, we choose the contour inside that neighborhood and change the variable from $z = z(p)$ to p as

$$\Phi_n(q) = \oint \frac{dp}{2\pi i} \frac{z(p)^n}{p - q}.$$

The contour of the transformed integral is a closed curve C that encircles $p = \infty$ but not q . This integral is nothing but the polynomial part of $z(q)^n$.

3 Toda hierarchy and difference Fay identity

3.1 Bilinear equations for tau function

The Toda hierarchy [41] has a discrete variable $s \in \mathbf{Z}$ (lattice coordinate) and two sets of continuous time variables $\mathbf{t} = (t_1, t_2, \dots)$, $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \dots)$. The tau function $\tau = \tau(s, \mathbf{t}, \bar{\mathbf{t}})$ satisfies an infinite number of Hirota equations

$$\begin{aligned} D_1 \bar{D}_1 \tau(s, \mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(s, \mathbf{t}, \bar{\mathbf{t}}) + 2\tau(s+1, \mathbf{t}, \bar{\mathbf{t}}) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}}) &= 0, \\ (D_2 + D_1^2) \tau(s+1, \mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(s, \mathbf{t}, \bar{\mathbf{t}}) &= 0, \\ (\bar{D}_2 + \bar{D}_1^2) \tau(s, \mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(s+1, \mathbf{t}, \bar{\mathbf{t}}) &= 0, \quad \dots, \end{aligned} \quad (3.1)$$

where D_n and \bar{D}_n are the Hirota bilinear operators associated with $\partial_n = \partial/\partial t_n$ and $\bar{\partial}_n = \partial/\partial \bar{t}_n$. A generating functional form of these Hirota equations is given by the bilinear equation

$$\begin{aligned} \oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(s', \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}') \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) \\ = \oint \frac{dz}{2\pi i} z^{s'-s} e^{\xi(\bar{\mathbf{t}}' - \bar{\mathbf{t}}, z^{-1})} \tau(s' + 1, \mathbf{t}', \bar{\mathbf{t}}' - [z]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [z]). \end{aligned} \quad (3.2)$$

Whereas the contour of integral on the left hand side is a sufficiently large circle $|z| = R$, that of the right hand side is a sufficiently small circle $|z| = R^{-1}$ around the origin (or one may understand both integrals as a purely algebraic operator extracting the coefficient of z^{-1}).

It is easy to from this bilinear equation that the Toda hierarchy contains the KP and modified KP hierarchies [30] as subsystems. For example, if \mathbf{t}' and s are specialized as $\bar{\mathbf{t}}' = \bar{\mathbf{t}}$ and $s' = s$, (3.2) reduces to the bilinear equation

$$\oint \frac{dz}{2\pi i} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(s, \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) = 0,$$

which is substantially the bilinear equation for the KP hierarchy. Thus the tau function of the Toda hierarchy, viewed as a function of \mathbf{t} , is a tau function of the KP hierarchy as well. The last equation can be generalized to the bilinear equations

$$\oint \frac{dz}{2\pi i} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(s', \mathbf{t}' - [z^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) = 0 \quad \text{for } s' \geq s, \quad (3.3)$$

which show that the tau function may be thought of as the tau function of the modified KP hierarchy with respect to \mathbf{t} and s . The same interpretation holds true for $\bar{\mathbf{t}}$ and s .

As in the case of the KP hierarchy, one can rewrite this bilinear equation into the Hirota form

$$\begin{aligned} & \sum_{n=0}^{\infty} h_n(-2\mathbf{a}) h_{n+s'-s+1}(\tilde{D}_{\mathbf{t}}) e^{\langle \mathbf{a}, D_{\mathbf{t}} \rangle + \langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle} \tau(s, \mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(s', \mathbf{t}, \bar{\mathbf{t}}) \\ &= \sum_{n=0}^{\infty} h_n(-2\bar{\mathbf{a}}) h_{n-s'+s-1}(\tilde{D}_{\bar{\mathbf{t}}}) e^{\langle \mathbf{a}, D_{\mathbf{t}} \rangle + \langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle} \tau(s-1, \mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(s'+1, \mathbf{t}, \bar{\mathbf{t}}), \end{aligned} \quad (3.4)$$

where $\mathbf{a} = (a_1, a_2, \dots)$ and $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots)$ are new variables, and $\tilde{D}_{\bar{\mathbf{t}}}$ and $\langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle$ are the counterparts of $\tilde{D}_{\mathbf{t}}$ and $\langle \mathbf{a}, D_{\mathbf{t}} \rangle$ for $\bar{\mathbf{t}}$, namely,

$$\tilde{D}_{\bar{\mathbf{t}}} = \left(\bar{D}_1, \frac{\bar{D}_2}{2}, \dots, \frac{\bar{D}_n}{n}, \dots \right), \quad \langle \bar{\mathbf{a}}, \bar{D}_{\bar{\mathbf{t}}} \rangle = \sum_{n=1}^{\infty} \bar{a}_n \bar{D}_n.$$

Expanded in powers of \mathbf{a} and $\bar{\mathbf{a}}$, (3.4) generates an infinite number of Hirota equations.

3.2 Difference Fay identities

We can derive three Fay-type identities [21, 42] from (3.2) by specializing $s', \mathbf{t}', \bar{\mathbf{t}}'$ as follows.

- (i) $s' = s + 1, \mathbf{t}' = \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}], \bar{\mathbf{t}}' = \bar{\mathbf{t}}.$
- (ii) $s' = s - 3, \mathbf{t}' = \mathbf{t}, \bar{\mathbf{t}}' = \bar{\mathbf{t}} + [\lambda] + [\mu],$
- (iii) $s' = s, \mathbf{t}' = \mathbf{t} + [\lambda^{-1}]. \bar{\mathbf{t}}' = \bar{\mathbf{t}} + [\mu].$

As we have shown in the case of the KP hierarchy, the exponential factors in the bilinear identity thereby become rational functions, and calculation of the contour integrals reduces to residue calculus.

Note that the role of $x = t_1$ in the KP hierarchy is now played by s . Unlike that case, however, we now shift s rather than differentiate the bilinear equation. For this reason, we call the following Fay-type identities “difference Fay identities”.

(i) The specialized bilinear equation becomes

$$\oint \frac{dz}{2\pi i} z \frac{\lambda\mu}{(z-\lambda)(z-\mu)} \tau(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) \\ = \oint \frac{dz}{2\pi i} z \tau(s+2, \mathbf{t}, \bar{\mathbf{t}} - [z]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [z]).$$

Residue calculus yields the first difference Fay identity

$$\tau(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]) \tau(s, \mathbf{t}, \bar{\mathbf{t}}) \\ - \frac{\lambda}{\lambda - \mu} \tau(s+1, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \\ + \frac{\mu}{\lambda - \mu} \tau(s+1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}}) = 0. \quad (3.5)$$

(ii) The specialized bilinear equation becomes

$$\oint \frac{dz}{2\pi i} z^{-3} \tau(s-3, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}} + [\lambda] + [\mu]) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) \\ = \oint \frac{dz}{2\pi i} z^{-3} \frac{z^2}{(z-\lambda)(z-\mu)} \tau(s-2, \mathbf{t}, \bar{\mathbf{t}} + [\lambda] + [\mu] - [z]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [z]).$$

Residue calculus yields the second difference Fay identity

$$\frac{1}{\lambda\mu} \tau(s-2, \mathbf{t}, \bar{\mathbf{t}} + [\lambda] + [\mu]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}}) \\ + \frac{1}{\lambda(\lambda - \mu)} \tau(s-2, \mathbf{t}, \bar{\mathbf{t}} + [\mu]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [\lambda]) \\ - \frac{1}{\mu(\lambda - \mu)} \tau(s-2, \mathbf{t}, \bar{\mathbf{t}} + [\lambda]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [\mu]) = 0. \quad (3.6)$$

(iii) The specialized bilinear equation becomes

$$\oint \frac{dz}{2\pi i} \frac{-\lambda}{z-\lambda} \tau(s, \mathbf{t} + [\lambda^{-1}] - [z^{-1}], \bar{\mathbf{t}} + [\mu]) \tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}}) \\ = \oint \frac{dz}{2\pi i} \frac{z}{z-\mu} \tau(s+1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}} + [\mu] - [z]) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [z]).$$

Residue calculus yields the third difference Fay identity:

$$\lambda \tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}} + [\mu]) \tau(s, \mathbf{t}, \bar{\mathbf{t}}) \\ - \lambda \tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu]) \\ + \mu \tau(s+1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [\mu]) = 0. \quad (3.7)$$

These difference Fay identities can be cast into different forms. The following are counterparts of (2.7):

$$\frac{\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t}, \bar{\mathbf{t}})}{\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}})} = \frac{\tau(s, \mathbf{t}, \bar{\mathbf{t}})}{(\lambda - \mu) \tau(s - 1, \mathbf{t}, \bar{\mathbf{t}})} \times \\ \times \left(\frac{\lambda \tau(s - 1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}})} - \frac{\mu \tau(s - 1, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t} + [\mu^{-1}], \bar{\mathbf{t}})} \right), \quad (3.8)$$

$$\frac{\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\lambda] + [\mu]) \tau(s, \mathbf{t}, \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\lambda]) \tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu])} = \frac{\tau(s, \mathbf{t}, \bar{\mathbf{t}})}{(\lambda - \mu) \tau(s + 1, \mathbf{t}, \bar{\mathbf{t}})} \times \\ \times \left(-\frac{\mu \tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [\lambda])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\lambda])} + \frac{\lambda \tau(s + 1, \mathbf{t}, \bar{\mathbf{t}} + [\mu])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu])} \right), \quad (3.9)$$

$$\frac{\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}} + [\mu]) \tau(s, \mathbf{t}, \bar{\mathbf{t}})}{\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu])} \\ = 1 - \frac{\mu}{\lambda} \frac{\tau(s + 1, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s - 1, \mathbf{t}, \bar{\mathbf{t}} + [\mu])}{\tau(s, \mathbf{t} + [\lambda^{-1}], \bar{\mathbf{t}}) \tau(s, \mathbf{t}, \bar{\mathbf{t}} + [\mu])}. \quad (3.10)$$

Similarly, the following equations are counterparts of (2.8):

$$\exp\left((e^{D(\lambda)} - 1)(e^{D(\mu)} - 1) \log \tau\right) \\ = \frac{1}{\lambda - \mu} \left(\lambda \exp\left((e^{D(\lambda)} - 1)(e^{-\partial_s} - 1) \log \tau\right) \right. \\ \left. - \mu \exp\left((e^{D(\mu)} - 1)(e^{-\partial_s} - 1) \log \tau\right) \right), \quad (3.11)$$

$$\exp\left((e^{\bar{D}(\lambda)} - 1)(e^{\bar{D}(\mu)} - 1) \log \tau\right) \\ = \frac{1}{\lambda - \mu} \left(-\mu \exp\left((e^{\bar{D}(\lambda)} - 1)(e^{\partial_s} - 1) \log \tau\right) \right. \\ \left. + \lambda \exp\left((e^{\bar{D}(\mu)} - 1)(e^{\partial_s} - 1) \log \tau\right) \right), \quad (3.12)$$

$$\exp\left((e^{D(\lambda)} - 1)(e^{\bar{D}(\mu)} - 1) \log \tau\right) \\ = 1 - \frac{\mu}{\lambda} \exp\left((e^{D(\lambda)} - 1)(e^{\partial_s} - 1) \log \tau + (e^{\bar{D}(\mu)} - 1)(e^{-\partial_s} - 1) \log \tau \right. \\ \left. - (e^{\partial_s} - 1)(e^{-\partial_s} - 1) \log \tau\right), \quad (3.13)$$

where

$$D(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_n, \quad \bar{D}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \bar{\partial}_n.$$

3.3 Auxiliary linear equations for wave functions

We now introduce the wave functions

$$\begin{aligned}\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(s, \mathbf{t} - [z^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} z^s e^{\xi(\mathbf{t}, z)}, \\ \bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} - [z])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} z^s e^{\xi(\bar{\mathbf{t}}, z^{-1})}\end{aligned}$$

and the duals

$$\begin{aligned}\Psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(s, \mathbf{t} + [z^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} z^{-s} e^{\xi(\mathbf{t}, z)}, \\ \bar{\Psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(s-1, \mathbf{t}, \bar{\mathbf{t}} + [z])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} z^{-s} e^{\xi(\bar{\mathbf{t}}, z^{-1})}.\end{aligned}$$

The bilinear equation (3.2) for the tau function becomes the bilinear equation

$$\oint \frac{dz}{2\pi i} \Psi(s', \mathbf{t}', \bar{\mathbf{t}}', z) \Psi^*(s, \mathbf{t}, \bar{\mathbf{t}}, z) = \oint \frac{dz}{2\pi i} \bar{\Psi}(s', \mathbf{t}', \bar{\mathbf{t}}', z) \bar{\Psi}^*(s, \mathbf{t}, \bar{\mathbf{t}}, z) \quad (3.14)$$

for these wave functions. From this bilinear equation, one can derive the auxiliary linear equations

$$(\partial_n - B_n)\Phi = 0, \quad (\bar{\partial}_n - \bar{B}_n)\Phi = 0 \quad (3.15)$$

for $\Phi = \Psi(s, \mathbf{t}, \bar{\mathbf{t}}, z), \bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, z)$. B_n and \bar{B}_n are difference operators of the form

$$\begin{aligned}B_n &= e^{n\partial_s} + b_{n,1}e^{(n-1)\partial_s} + \cdots + b_{n,n}, \\ \bar{B}_n &= \bar{b}_{n,0}e^{-n\partial_s} + \cdots + \bar{b}_{n,n-1}e^{-\partial_s},\end{aligned}$$

and satisfy the Zakharov-Shabat equations

$$\begin{aligned}[\partial_m - B_n, \partial_n - B_n] &= 0, \quad [\bar{\partial}_m - \bar{B}_n, \bar{\partial}_n - \bar{B}_n] = 0, \\ [\partial_m - B_m, \bar{\partial}_n - \bar{B}_n] &= 0.\end{aligned} \quad (3.16)$$

We can construct two Lax operators of the form

$$L = e^{\partial_s} + \sum_{j=0}^{\infty} u_j e^{-j\partial_s}, \quad \bar{L} = \sum_{j=1}^{\infty} \bar{u}_j e^{j\partial_s}$$

such that the Zakharov-Shabat operators are expressed as

$$B_n = (L^n)_{\geq 0}, \quad \bar{B}_n = (\bar{L}^n)_{< 0},$$

where $(\)_{\geq 0}$ and $(\)_{< 0}$ are the projection onto nonnegative and negative powers of e^{∂_s} , respectively. The Lax operators satisfy the Lax equations

$$\begin{aligned}\partial_n(L) &= [B_n, L], \quad \partial_n(\bar{L}) = [B_n, \bar{L}], \\ \bar{\partial}_n(L) &= [\bar{B}_n, L], \quad \bar{\partial}_n(\bar{L}) = [\bar{B}_n, \bar{L}].\end{aligned} \quad (3.17)$$

3.4 Difference Fay identities and auxiliary linear equations

We now rewrite the difference Fay identities (3.8) – (3.10) in the language of the wave functions. This leads to a generating functional expression of the auxiliary linear problem.

Let us first consider (3.8). We shift the time variables as $\mathbf{t} \rightarrow \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}]$ and multiply both hand sides by $(\lambda - \mu)e^{\xi(\mathbf{t}, \mu)}/\tau(s+1, \mathbf{t}, \bar{\mathbf{t}})\tau(s, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}})$. This yields the equation

$$\begin{aligned} & (\lambda - \mu)e^{\xi(\mathbf{t}, \mu)}e^{-D(\lambda)}\frac{\tau(s, \mathbf{t} - [\mu^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} \\ &= \lambda\frac{\tau(s+1, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}})/\tau(s+1, \mathbf{t}, \bar{\mathbf{t}})}{\tau(s, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}})/\tau(s, \mathbf{t}, \bar{\mathbf{t}})}\mu^se^{\xi(\mathbf{t}, \mu)}\frac{\tau(s, \mathbf{t} - [\mu^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} \\ & \quad - \mu^{s+1}e^{\xi(\mathbf{t}, \mu)}\frac{\tau(s+1, \mathbf{t} - [\mu^{-1}], \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})}. \end{aligned}$$

Noting that the commutator of $e^{-D(\lambda)}$ and $e^{\xi(\mathbf{t}, \mu)}$ yields the factor $1 - \mu/\lambda$, we can rewrite the last equation as

$$\lambda e^{-D(\lambda)}\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) = \frac{\Psi(s+1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) - \Psi(s+1, \mathbf{t}, \bar{\mathbf{t}}, \mu). \quad (3.18)$$

Let us now consider (3.9). We now shift $\bar{\mathbf{t}}$ and s as $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - [\lambda] - [\mu]$ and $s \rightarrow s+2$, and multiply both hand side by $\lambda(\lambda - \mu)\mu^se^{\xi(\bar{\mathbf{t}}, \mu^{-1})}/\tau(s, \mathbf{t}, \bar{\mathbf{t}})\tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\lambda])$. This yields the equation

$$\begin{aligned} & -\frac{\lambda - \mu}{\mu}\mu^se^{\xi(\bar{\mathbf{t}}, \mu^{-1})}e^{-\bar{D}(\lambda)}\frac{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} - [\mu])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} \\ &= \mu^se^{\xi(\bar{\mathbf{t}}, \mu^{-1})}\frac{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} - [\mu^{-1}])}{\tau(s, \mathbf{t}, \bar{\mathbf{t}})} - \lambda\frac{\tau(s+1, \mathbf{t}, \bar{\mathbf{t}} - [\lambda])/\tau(s, \mathbf{t}, \bar{\mathbf{t}})}{\tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\lambda])/\tau(s-1, \mathbf{t}, \bar{\mathbf{t}})} \times \\ & \quad \times \mu^{s-1}e^{\xi(\mathbf{t}, \mu)}\frac{\tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\mu])}{\tau(s-1, \mathbf{t}, \bar{\mathbf{t}})}, \end{aligned}$$

which can be cast into the form

$$e^{-\bar{D}(\lambda)}\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) = \bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) - \frac{\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \mu). \quad (3.19)$$

In the same way, we can derive two more linear equations for the wave functions from (3.7). Firstly, if we shift the time variables as $\mathbf{t} \rightarrow \mathbf{t} - [\lambda^{-1}]$ and $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - [\mu]$ and multiply both hand side by $\lambda^se^{\xi(\mathbf{t}, \lambda)}/\tau(s, \mathbf{t}, \bar{\mathbf{t}})\tau(s, \mathbf{t}, \bar{\mathbf{t}} - [\mu])$, we obtain the equation

$$e^{-\bar{D}(\mu)}\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda) = \Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda) - \frac{\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu)}{\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \mu)}\Psi(s-1, \mathbf{t}, \bar{\mathbf{t}}, \lambda).$$

It will be better to exchange λ and μ as

$$e^{-\bar{D}(\lambda)}\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) = \Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) - \frac{\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}\Psi(s-1, \mathbf{t}, \bar{\mathbf{t}}, \mu), \quad (3.20)$$

because the outcome takes the same form as (3.19). Secondly, if we shift $\mathbf{t}, \bar{\mathbf{t}}$ and s as $\mathbf{t} \rightarrow \mathbf{t} - [\lambda^{-1}]$, $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - [\mu]$, $s \rightarrow s+1$ and multiply both hand side by $\lambda\mu^s e^{\xi(\bar{\mathbf{t}}, \mu^{-1})} / \tau(s+1, \mathbf{t}, \bar{\mathbf{t}}) \tau(s, \mathbf{t} - [\lambda^{-1}], \bar{\mathbf{t}})$, we are led to the equation

$$\lambda e^{-D(\lambda)}\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) = \frac{\Psi(s+1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu) - \bar{\Psi}(s+1, \mathbf{t}, \bar{\mathbf{t}}, \mu). \quad (3.21)$$

(3.18) – (3.21) show that the linear equations

$$\begin{aligned} \lambda e^{-D(\lambda)}\Phi &= \frac{\Psi(s+1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}\Phi - e^{\partial_s}\Phi, \\ e^{-\bar{D}(\lambda)}\Phi &= \Phi - \frac{\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}e^{-\partial_s}\Phi \end{aligned}$$

are satisfied by both $\Phi = \Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \mu)$ and $\Phi = \bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \mu)$. Expanded in powers of λ , they generate an infinite number of linear equations of the form

$$h_n(-\tilde{\partial}_{\mathbf{t}})\Phi = v_n\Phi, \quad h_n(-\tilde{\partial}_{\bar{\mathbf{t}}})\Phi = -\bar{v}_n e^{-\partial_s}\Phi, \quad (3.22)$$

where $\tilde{\partial}_{\mathbf{t}}$ is the same notation as defined for the KP hierarchy, $\tilde{\partial}_{\bar{\mathbf{t}}}$ is similarly defined as

$$\tilde{\partial}_{\bar{\mathbf{t}}} = \left(\bar{\partial}_1, \frac{\bar{\partial}_2}{2}, \dots, \frac{\bar{\partial}_n}{n}, \dots \right),$$

and $v_j = v_j(s, \mathbf{t}, \bar{\mathbf{t}})$ and $\bar{v}_j = \bar{v}_j(s, \mathbf{t}, \bar{\mathbf{t}})$ are the Laurent coefficients of the Ψ -quotients in the foregoing linear equations, namely,

$$\frac{\Psi(s+1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\Psi(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)} = \lambda + \sum_{n=1}^{\infty} v_n \lambda^{1-n}, \quad \frac{\bar{\Psi}(s, \mathbf{t}, \bar{\mathbf{t}}, \lambda)}{\bar{\Psi}(s-1, \mathbf{t}, \bar{\mathbf{t}}, \lambda)} = \sum_{n=1}^{\infty} \bar{v}_n \lambda^n.$$

Note that the lowest ($n=1$) equations of (3.22) read

$$\partial_1\Phi = (e^{\partial_s} - v_1)\Phi, \quad \bar{\partial}_1\Phi = \bar{v}_1 e^{-\partial_s}\Phi. \quad (3.23)$$

As we have illustrated in the case of the KP hierarchy, one can derive the usual evolutionary form (3.15) of auxiliary linear equations from these equations (3.22) and (3.23). On the basis of this fact, Teo [42] proved that the difference Fay identities are equivalent to the full system of the Toda hierarchy.

3.5 Dispersionless Hirota equations and Hamilton-Jacobi equations

The quasi-classical ansatz (2.22) for the tau function of the KP hierarchy can be readily generalized to the Toda hierarchy [43]. Namely, we allow the tau

function to depend on \hbar as $\tau = \tau(\hbar, s, \mathbf{t}, \bar{\mathbf{t}})$, and assume that the rescaled tau function $\tau_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}) = \tau(\hbar, \hbar^{-1}s, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}})$ behave as

$$\tau_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}) = \exp\left(\hbar^{-2}F(s, \mathbf{t}, \bar{\mathbf{t}}) + O(\hbar^{-1})\right) \quad (3.24)$$

in the limit of $\hbar \rightarrow 0$. Note that the rescaling changes the lattice spacing from 1 to \hbar , eventually tending to 0 in the quasi-classical limit. The discrete variable s in $\tau(\hbar, s, \mathbf{t}, \bar{\mathbf{t}})$ is thereby replaced by a continuous variable in $F(s, \mathbf{t}, \bar{\mathbf{t}})$.

Under this quasi-classical ansatz, we can derive the following three dispersionless Hirota equations [20, 21, 14] for the F function $F = F(s, \mathbf{t}, \bar{\mathbf{t}})$ from the difference Fay identities:

$$\begin{aligned} e^{D(\lambda)D(\mu)F} &= \frac{\lambda e^{-D(\lambda)\partial_s F} - \mu e^{-D(\mu)\partial_s F}}{\lambda - \mu}, \\ e^{\bar{D}(\lambda)\bar{D}(\mu)F} &= \frac{-\mu e^{\bar{D}(\lambda)\partial_s F} + \lambda e^{\bar{D}(\mu)\partial_s F}}{\lambda - \mu}, \\ e^{D(\lambda)\bar{D}(\mu)F} &= 1 - \frac{\mu}{\lambda} e^{(D(\lambda) - \bar{D}(\mu) + \partial_s)\partial_s F}. \end{aligned} \quad (3.25)$$

It will be better to rewrite the second equation as

$$e^{\bar{D}(\lambda)\bar{D}(\mu)F} = \frac{\lambda^{-1} e^{\bar{D}(\lambda)\partial_s F} - \mu^{-1} e^{\bar{D}(\mu)\partial_s F}}{\lambda^{-1} - \mu^{-1}},$$

by which the symmetry between \mathbf{t} and $\bar{\mathbf{t}}$ becomes manifest. Moreover, defining the S functions as

$$\begin{aligned} S(z) &= \xi(\mathbf{t}, z) + s \log z - D(z)F, \\ \bar{S}(z) &= \xi(\bar{\mathbf{t}}, z^{-1}) + s \log z + \partial_s F - \bar{D}(z)F, \end{aligned}$$

we can rewrite (3.25) as

$$\begin{aligned} e^{D(\lambda)D(\mu)F} &= \frac{e^{\partial_s S(\lambda)} - e^{\partial_s S(\mu)}}{\lambda - \mu}, \\ e^{\bar{D}(\lambda)\bar{D}(\mu)F} &= e^{\partial_s^2 F} \frac{e^{-\partial_s \bar{S}(\lambda)} - e^{-\partial_s \bar{S}(\mu)}}{\lambda^{-1} - \mu^{-1}}, \\ e^{D(\lambda)\bar{D}(\mu)F} &= 1 - e^{-\partial_s S(\lambda)} e^{\partial_s \bar{S}(\mu)}. \end{aligned} \quad (3.26)$$

The S functions are actually the phase functions in the WKB ansatz

$$\begin{aligned} \Psi_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \exp\left(\hbar^{-1}S(z) + O(\hbar^0)\right), \\ \bar{\Psi}_{\hbar}(s, \mathbf{t}, \bar{\mathbf{t}}, z) &= \exp\left(\hbar^{-1}\bar{S}(z) + O(\hbar^0)\right) \end{aligned} \quad (3.27)$$

for the rescaled wave functions. The auxiliary linear equations thereby yield the Hamilton-Jacobi equations

$$\partial_n S = \mathcal{B}_n(e^{\partial_s S}), \quad \bar{\partial}_n S = \bar{\mathcal{B}}_n(e^{\partial_s S}) \quad (3.28)$$

for $S = S(z), \bar{S}(z)$. Here $\mathcal{B}_n(P)$ and $\bar{\mathcal{B}}_n(P)$ are classical counterparts of $B_n = B_n(e^{\partial_s})$ and $\bar{B}_n = \bar{B}_n(e^{\partial_s})$:

$$\mathcal{B}_n(P) = \lim_{\hbar \rightarrow 0} B_{\hbar,n}(P), \quad \bar{\mathcal{B}}_n(P) = \lim_{\hbar \rightarrow 0} \bar{B}_{\hbar,n}(P).$$

The dispersionless Hirota equations in the form of (3.26) turn out to be equivalent to these Hamilton-Jacobi equations, hence to the dispersionless Toda hierarchy itself [14, 24, 25].

4 BKP hierarchy and differential Fay identities

4.1 Bilinear equations for tau functions

In this section, we mostly consider the two-component version [44, 45] of the BKP hierarchy [46]. This system contains two copies of the one-component BKP hierarchy as subsystems. The situation is thus somewhat similar to the Toda hierarchy, which may be thought of as the charged two-component KP hierarchy as well [41]. This is a reason why we are interested in the two-component case rather than the one-component BKP hierarchy.

The two-component BKP hierarchy has two sets of time variables $\mathbf{t} = (t_1, t_3, \dots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_3, \dots)$ indexed by odd positive integers. The tau function $\tau = \tau(\mathbf{t}, \bar{\mathbf{t}})$ satisfies the bilinear equation

$$\oint \frac{dz}{2\pi iz} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(\mathbf{t}' - 2[z^{-1}], \bar{\mathbf{t}}') \tau(\mathbf{t} + 2[z^{-1}], \bar{\mathbf{t}}) \\ = \oint \frac{dz}{2\pi iz} e^{\xi(\bar{\mathbf{t}}' - \bar{\mathbf{t}}, z)} \tau(\mathbf{t}', \bar{\mathbf{t}}' - 2[z^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[z^{-1}]). \quad (4.1)$$

The integrals on both hand sides are understood to be a contour integral along a sufficiently large circle $|z| = R$ (or just a formal algebraic operator). The other notations are as follows:

$$\xi(\mathbf{t}, z) = \sum_{n=0}^{\infty} t_{2n+1} z^{2n+1}, \quad [\alpha] = \left(\alpha, \frac{\alpha^3}{3}, \dots, \frac{\alpha^{2n+1}}{2n+1}, \dots \right).$$

If \mathbf{t} is specialized as $\bar{\mathbf{t}}' = \bar{\mathbf{t}}$, the bilinear equation (4.1) reduces to

$$\oint \frac{dz}{2\pi iz} e^{\xi(\mathbf{t}' - \mathbf{t}, z)} \tau(\mathbf{t}' - 2[z^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t} + 2[z^{-1}], \bar{\mathbf{t}}) = \tau(\mathbf{t}', \bar{\mathbf{t}}) \tau(\mathbf{t}, \bar{\mathbf{t}}). \quad (4.2)$$

Similarly, specializing \mathbf{t}' as $\mathbf{t}' = \mathbf{t}$ yields

$$\oint \frac{dz}{2\pi iz} e^{\xi(\bar{\mathbf{t}}' - \bar{\mathbf{t}}, z)} \tau(\mathbf{t}, \bar{\mathbf{t}}' - 2[z^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[z^{-1}]) = \tau(\mathbf{t}, \bar{\mathbf{t}}') \tau(\mathbf{t}, \bar{\mathbf{t}}). \quad (4.3)$$

These equations coincide with the bilinear equation of the one-component BKP hierarchy. Thus the tau function of the two-component BKP hierarchy may

be thought of as the tau function of the one-component BKP hierarchy with respect to both \mathbf{t} and $\bar{\mathbf{t}}$.

(4.1) can be converted to the Hirota form

$$\begin{aligned} \sum_{n=0}^{\infty} h_n(-2\mathbf{a})h_n(2\tilde{D}_{\mathbf{t}})e^{\langle \mathbf{a}, D_{\mathbf{t}} \rangle + \langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle} \tau(\mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(\mathbf{t}, \bar{\mathbf{t}}) \\ = \sum_{n=0}^{\infty} h_n(-2\bar{\mathbf{a}})h_n(2\tilde{D}_{\bar{\mathbf{t}}})e^{\langle \mathbf{a}, D_{\mathbf{t}} \rangle + \langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle} \tau(\mathbf{t}, \bar{\mathbf{t}}) \cdot \tau(\mathbf{t}, \bar{\mathbf{t}}), \end{aligned} \quad (4.4)$$

where $\mathbf{a} = (a_1, a_3, \dots)$ and $\bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_3, \dots)$ are new variables, $D_{\mathbf{t}}, \tilde{D}_{\bar{\mathbf{t}}}$, and $\langle \mathbf{a}, D_{\mathbf{t}} \rangle, \langle \bar{\mathbf{a}}, D_{\bar{\mathbf{t}}} \rangle$ are operators defined as

$$\begin{aligned} \tilde{D}_{\mathbf{t}} &= \left(D_1, \frac{D_3}{3}, \dots, \frac{D_{2n+1}}{2n+1}, \dots \right), \quad \langle \mathbf{a}, D_{\mathbf{t}} \rangle = \sum_{n=0}^{\infty} a_{2n+1} D_{2n+1}, \\ \tilde{D}_{\bar{\mathbf{t}}} &= \left(\bar{D}_1, \frac{\bar{D}_3}{3}, \dots, \frac{\bar{D}_{2n+1}}{2n+1}, \dots \right), \quad \langle \bar{\mathbf{a}}, \bar{D}_{\bar{\mathbf{t}}} \rangle = \sum_{n=0}^{\infty} \bar{a}_{2n+1} \bar{D}_{2n+1}, \end{aligned}$$

and $h_n(\mathbf{t})$ are defined by the generating function

$$e^{\xi(\mathbf{t}, z)} = \exp\left(\sum_{n=0}^{\infty} t_{2n+1} z^{2n+1}\right) = \sum_{n=0}^{\infty} h_n(\mathbf{t}) z^n.$$

4.2 Differential Fay identities

We can derive four differential Fay identities [47] from (4.1) as follows.

- (i) Differentiate the bilinear equation by t'_1 and specialize $\mathbf{t}', \bar{\mathbf{t}}'$ as $\mathbf{t}' = \mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}]$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}}$.
- (ii) Differentiate the bilinear equation by \bar{t}'_1 and specialize $\mathbf{t}', \bar{\mathbf{t}}'$ as $\mathbf{t}' = \mathbf{t}$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}} + 2[\lambda^{-1}] + 2[\mu^{-1}]$.
- (iii) Differentiate the bilinear equation by t'_1 and specialize $\mathbf{t}, \bar{\mathbf{t}}'$ as $\mathbf{t}' = \mathbf{t} + 2[\lambda^{-1}]$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}} + 2[\mu^{-1}]$.
- (iv) Differentiate the bilinear equation by \bar{t}'_1 and specialize $\mathbf{t}, \bar{\mathbf{t}}'$ as $\mathbf{t}' = \mathbf{t} + 2[\lambda^{-1}]$, $\bar{\mathbf{t}}' = \bar{\mathbf{t}} + 2[\mu^{-1}]$.

Note that the exponential factors in the bilinear equations become rational functions by virtue of the identity

$$\sum_{n=0}^{\infty} \frac{2}{2n+1} \left(\frac{z}{w}\right)^{2n+1} = -\log \frac{1-z/w}{1+z/w}.$$

For example, in the case of (i),

$$e^{\xi(\mathbf{t}' - \mathbf{t}, z)} = \frac{(z + \lambda)(z + \mu)}{(z - \lambda)(z - \mu)}, \quad e^{\xi(\bar{\mathbf{t}}' - \bar{\mathbf{t}}, z)} = 1.$$

We can thus calculate the contour integrals by residue calculus.

(i) After differentiating by t'_1 and specializing \mathbf{t}' , the bilinear equation becomes

$$\begin{aligned} & \oint \frac{dz}{2\pi iz} \frac{(z+\lambda)(z+\mu)}{(z-\lambda)(z-\mu)} \left(z\tau(\mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}] - 2[z^{-1}], \bar{\mathbf{t}}) \right. \\ & \quad \left. + (\partial_1 \tau)(\mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}], \bar{\mathbf{t}}) \right) \tau(\mathbf{t} + 2[z^{-1}], \bar{\mathbf{t}}) \\ & = \oint \frac{dz}{2\pi iz} (\partial_1 \tau)(\mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}], \bar{\mathbf{t}} - 2[z^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[z^{-1}]). \end{aligned}$$

By residue calculus, we obtain the first differential Fay identity

$$\begin{aligned} & \left(\lambda + \mu - \partial_1 \log \frac{\tau(\mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t}, \bar{\mathbf{t}})} \right) \times \\ & \quad \times \frac{\tau(\mathbf{t} + 2[\lambda^{-1}] + 2[\mu^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t}, \bar{\mathbf{t}})}{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t} + 2[\mu^{-1}], \bar{\mathbf{t}})} \\ & = \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - \partial_1 \log \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t} + 2[\mu^{-1}], \bar{\mathbf{t}})} \right). \quad (4.5) \end{aligned}$$

(ii) This case is substantially the same as (i), only the role of \mathbf{t} and $\bar{\mathbf{t}}$ being exchanged. We thus obtain the second differential Fay identity

$$\begin{aligned} & \left(\lambda + \mu - \bar{\partial}_1 \log \frac{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\lambda^{-1}] + 2[\mu^{-1}])}{\tau(\mathbf{t}, \bar{\mathbf{t}})} \right) \times \\ & \quad \times \frac{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\lambda^{-1}] + 2[\mu^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}})}{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\lambda^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\mu^{-1}])} \\ & = \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - \bar{\partial}_1 \log \frac{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\lambda^{-1}])}{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\mu^{-1}])} \right). \quad (4.6) \end{aligned}$$

(iii) After differentiation and specialization, the bilinear equation becomes

$$\begin{aligned} & \oint \frac{dz}{2\pi iz} \frac{(z+\lambda)(z+\mu)}{(z-\lambda)(z-\mu)} \left(z\tau(\mathbf{t} + 2[\lambda^{-1}] - 2[z^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}]) \right. \\ & \quad \left. + (\partial_1 \tau)(\mathbf{t} + 2[\lambda^{-1}] - 2[z^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}]) \right) \tau(\mathbf{t} + 2[z^{-1}], \bar{\mathbf{t}}) \\ & = \oint \frac{dz}{2\pi iz} \frac{z+\mu}{z-\mu} (\partial_1 \tau)(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}] - 2[z^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[z^{-1}]). \end{aligned}$$

By residue calculus, we obtain the third differential Fay identity

$$\begin{aligned} & \left(\lambda - \partial_1 \log \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}])}{\tau(\mathbf{t}, \bar{\mathbf{t}})} \right) \times \\ & \quad \times \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}]) \tau(\mathbf{t}, \bar{\mathbf{t}})}{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\mu^{-1}])} \\ & = \lambda - \partial_1 \log \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t}, \bar{\mathbf{t}} + [\mu^{-1}])}. \quad (4.7) \end{aligned}$$

(iv) This case is parallel to (iii), and leads to the fourth differential Fay identity

$$\begin{aligned} & \left(\mu - \bar{\partial}_1 \log \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}} + [\mu^{-1}])}{\tau(\mathbf{t}, \bar{\mathbf{t}})} \right) \times \\ & \quad \times \frac{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}} + 2[\mu^{-1}])\tau(\mathbf{t}, \bar{\mathbf{t}})}{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}})\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\mu^{-1}])} \\ & = \mu - \bar{\partial}_1 \log \frac{\tau(\mathbf{t}, \bar{\mathbf{t}} + 2[\mu^{-1}])}{\tau(\mathbf{t} + 2[\lambda^{-1}], \bar{\mathbf{t}})}. \end{aligned} \quad (4.8)$$

We can use the BKP version

$$D(z) = \sum_{n=0}^{\infty} \frac{z^{-2n-1}}{2n+1} \partial_{2n+1}, \quad \bar{D}(z) = \sum_{n=0}^{\infty} \frac{z^{-2n-1}}{2n+1} \bar{\partial}_{2n+1}$$

of the $D(z)$ operator to rewrite the foregoing differential Fay identities as follows:

$$\begin{aligned} & \left(\lambda + \mu - \partial_1 (e^{2D(\lambda)+2D(\mu)} - 1) \log \tau \right) \times \\ & \quad \times \exp \left((e^{2D(\lambda)} - 1)(e^{2D(\mu)} - 1) \log \tau \right) \\ & = \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - \partial_1 (e^{2D(\lambda)} - e^{2D(\mu)}) \log \tau \right), \end{aligned} \quad (4.9)$$

$$\begin{aligned} & \left(\lambda + \mu - \bar{\partial}_1 (e^{2\bar{D}(\lambda)+2\bar{D}(\mu)} - 1) \log \tau \right) \times \\ & \quad \times \exp \left((e^{2\bar{D}(\lambda)} - 1)(e^{2\bar{D}(\mu)} - 1) \log \tau \right) \\ & = \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - \bar{\partial}_1 (e^{2\bar{D}(\lambda)} - e^{2\bar{D}(\mu)}) \log \tau \right), \end{aligned} \quad (4.10)$$

$$\begin{aligned} & \left(\lambda - \partial_1 (e^{2D(\lambda)+2\bar{D}(\mu)} - 1) \log \tau \right) \times \\ & \quad \times \exp \left((e^{2D(\lambda)} - 1)(e^{2\bar{D}(\mu)} - 1) \log \tau \right) \\ & = \lambda - \partial_1 (e^{2D(\lambda)} - e^{2\bar{D}(\mu)}) \log \tau, \end{aligned} \quad (4.11)$$

$$\begin{aligned} & \left(\mu - \bar{\partial}_1 (e^{2D(\lambda)+2\bar{D}(\mu)} - 1) \log \tau \right) \times \\ & \quad \times \exp \left((e^{2D(\lambda)} - 1)(e^{2\bar{D}(\mu)} - 1) \log \tau \right) \\ & = \mu - \bar{\partial}_1 (e^{2\bar{D}(\mu)} - e^{2D(\lambda)}) \log \tau. \end{aligned} \quad (4.12)$$

4.3 Auxiliary linear equations

We now introduce the two wave functions

$$\begin{aligned}\Psi(\mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(\mathbf{t} - 2[z^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t}, \bar{\mathbf{t}})} e^{\xi(\mathbf{t}, z)}, \\ \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, z) &= \frac{\tau(\mathbf{t}, \bar{\mathbf{t}} - 2[z^{-1}])}{\tau(\mathbf{t}, \bar{\mathbf{t}})} e^{\xi(\bar{\mathbf{t}}, z)}.\end{aligned}$$

Their duals are given by $\Psi(\mathbf{t}, \bar{\mathbf{t}}, -z)$ and $\bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, -z)$. The bilinear equation (4.1) for the tau function yields the bilinear equation

$$\oint \frac{dz}{2\pi iz} \Psi(\mathbf{t}', \bar{\mathbf{t}}', z) \Psi(\mathbf{t}, \bar{\mathbf{t}}, -z) = \oint \frac{dz}{2\pi iz} \bar{\Psi}(\mathbf{t}', \bar{\mathbf{t}}', z) \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, -z) \quad (4.13)$$

for these wave functions. From this bilinear equations, one can derive an infinite number of auxiliary linear equations of the form

$$(\partial_{2n+1} - B_{2n+1})\Phi = 0, \quad (\bar{\partial}_{2n+1} - \bar{B}_{2n+1})\Phi = 0 \quad (4.14)$$

for $n = 0, 1, \dots$ and

$$(\partial_1 \bar{\partial}_1 - u)\Phi = 0 \quad (4.15)$$

that hold for both $\Phi = \Psi(\mathbf{t}, \bar{\mathbf{t}}, z)$ and $\Phi = \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, z)$. $B_{2n+1} = B_{2n+1}(\partial_1)$ and $\bar{B}_{2n+1} = \bar{B}_{2n+1}(\bar{\partial}_1)$ are differential operators with respect to t_1 and \bar{t}_1 of the form

$$\begin{aligned}B_{2n+1} &= \partial_1^{2n+1} + b_{2n+1,2} \partial_1^{2n-1} + \dots + b_{2n+1,2n} \partial_1, \\ \bar{B}_{2n+1} &= \bar{\partial}_1^{2n+1} + \bar{b}_{2n+1,2} \bar{\partial}_1^{2n-1} + \dots + \bar{b}_{2n+1,2n} \bar{\partial}_1.\end{aligned}$$

Note that they do not have a 0-th order term. This is a characteristic of the operators that show up in the auxiliary linear problem of the one-component BKP hierarchy [46]. The first two sets of equations (4.14) are thus nothing but the auxiliary linear problem of the underlying one-component BKP hierarchies. The third equation (4.15) may be thought of as the two-dimensional “integrable Schrödinger equation” associated with the Novikov-Veselov equation [48]. The potential u is given by

$$u = -2\partial_1 \bar{\partial}_1 \log \tau. \quad (4.16)$$

In this respect, the two-component BKP hierarchy is also referred to as the Novikov-Veselov hierarchy. t_1 and \bar{t}_1 play the role of spatial variables therein. Auxiliary linear equations of this type play a central role in Krichever’s recent work [49] on a Schottky-type problem on Prym varieties [50].

The Lax formalism of this hierarchy exhibits new features because of the presence of two spatial dimensions. For example, the Zakharov-Shabat operators of the same type satisfy the usual zero-curvature equations

$$\begin{aligned}[\partial_{2m+1} - B_{2m+1}, \partial_{2n+1} - B_{2n+1}] &= 0, \\ [\bar{\partial}_{2m+1} - \bar{B}_{2m+1}, \bar{\partial}_{2n+1} - \bar{B}_{2n+1}] &= 0,\end{aligned} \quad (4.17)$$

but the equation for the pair of B_{2m+1} and \bar{B}_{2n+1} has an extra term on the right hand side as

$$[\partial_{2m+1} - B_{2m+1}, \bar{\partial}_{2n+1} - \bar{B}_{2n+1}] = D_{mn}(\partial_1, \bar{\partial}_1)(\partial_1 \bar{\partial}_1 - u), \quad (4.18)$$

where $D_{mn}(\partial_1, \bar{\partial}_1)$ is a differential operator with respect to both t_1 and \bar{t}_1 [48, 49].

4.4 Differential Fay identities and auxiliary linear equations

As we have demonstrated for the cases of the KP and Toda hierarchies, the differential Fay identities in the present case, too, give a generating functional expression of the auxiliary linear problem. The situation, however, turns out to be more complicated than the previous two cases.

Let us first consider (4.5) and (4.6). As regards (4.5), we first shift \mathbf{t} as $\mathbf{t} \rightarrow \mathbf{t} - 2[\lambda^{-1}] - 2[\mu^{-1}]$. The equation thereby changes to

$$\begin{aligned} & \left(\lambda + \mu + \partial_1 \log \frac{\tau(\mathbf{t} - 2[\lambda^{-1}] - 2[\mu^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t}, \bar{\mathbf{t}})} \right) \times \\ & \quad \times \frac{\tau(\mathbf{t} - 2[\lambda^{-1}] - 2[\mu^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t}, \bar{\mathbf{t}})}{\tau(\mathbf{t} - 2[\lambda^{-1}], \bar{\mathbf{t}}) \tau(\mathbf{t} - 2[\mu^{-1}], \bar{\mathbf{t}})} \\ & \quad = \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu + \partial_1 \log \frac{\tau(\mathbf{t} - 2[\lambda^{-1}], \bar{\mathbf{t}})}{\tau(\mathbf{t} - 2[\mu^{-1}], \bar{\mathbf{t}})} \right). \end{aligned}$$

We then multiply both hand sides by $e^{\xi(\mathbf{t}, \mu)} \tau(\mathbf{t} - 2[\mu^{-1}], \bar{\mathbf{t}}) / \tau(\mathbf{t}, \bar{\mathbf{t}})$. After some algebra, we obtain the equation

$$\begin{aligned} & (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) + \partial_1) e^{-2D(\lambda)} \Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ & \quad = (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) - \partial_1) \Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.19)$$

In much the same way, shifting $\bar{\mathbf{t}}$ as $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - 2[\lambda^{-1}] - 2[\mu^{-1}]$ in (4.6) and multiplying it by $e^{\xi(\bar{\mathbf{t}}, \mu)} \tau(\mathbf{t}, \bar{\mathbf{t}} - 2[\mu^{-1}]) / \tau(\mathbf{t}, \bar{\mathbf{t}})$, we obtain the equation

$$\begin{aligned} & (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda) + \bar{\partial}_1) e^{-2\bar{D}(\lambda)} \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ & \quad = (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda) - \bar{\partial}_1) \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.20)$$

We can rewrite (4.7) and (4.8) in a similar way. As regards (4.7), we shift both \mathbf{t} and $\bar{\mathbf{t}}$ as $\mathbf{t} \rightarrow \mathbf{t} - 2[\lambda^{-1}]$ and $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - 2[\mu^{-1}]$, and multiply both hand sides by $e^{\xi(\bar{\mathbf{t}}, \mu)} \tau(\mathbf{t}, \bar{\mathbf{t}} - 2[\mu^{-1}]) / \tau(\mathbf{t}, \bar{\mathbf{t}})$. This leads to the equation

$$\begin{aligned} & (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) + \partial_1) e^{-2D(\lambda)} \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ & \quad = (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) - \partial_1) \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.21)$$

Note that this equation has the same form as (4.5). Anticipating a similar result, we now consider (4.8). If we shift \mathbf{t} and $\bar{\mathbf{t}}$ as $\mathbf{t} \rightarrow \mathbf{t} - 2[\lambda^{-1}]$ and $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - 2[\mu^{-1}]$

and multiply this equation by $e^{\xi(\mathbf{t}, \lambda)} \tau(\mathbf{t} - 2[\lambda^{-1}], \bar{\mathbf{t}}) / \tau(\mathbf{t}, \bar{\mathbf{t}})$, the outcome is the equation

$$(\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) + \bar{\partial}_1) e^{-2\bar{D}(\mu)} \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) = (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) - \bar{\partial}_1) \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda).$$

which is almost what we want. By exchanging λ and μ , we obtain the equation

$$\begin{aligned} (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda) + \bar{\partial}_1) e^{-2\bar{D}(\lambda)} \Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ = (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda) - \bar{\partial}_1) \Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.22)$$

As anticipated, this equation has the same form as (4.20).

These results show that $\Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu)$ and $\bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu)$ satisfy the same linear equations

$$\begin{aligned} (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) + \partial_1) e^{-2D(\lambda)} \Phi &= (\partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) - \partial_1) \Phi, \\ (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) + \bar{\partial}_1) e^{-2\bar{D}(\mu)} \Phi &= (\bar{\partial}_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) - \bar{\partial}_1) \Phi. \end{aligned}$$

Expanded in powers of λ , they give rise to an infinite number of equations of the form

$$\begin{aligned} \left(h_{n+1}(-2\tilde{\partial}_{\mathbf{t}}) + \sum_{m=1}^{n-1} v_{m+1} h_{n-m}(-2\tilde{\partial}_{\mathbf{t}}) + \partial_1 h_n(-2\tilde{\partial}_{\mathbf{t}}) \right) \Phi &= 0, \\ \left(h_{n+1}(-2\tilde{\partial}_{\bar{\mathbf{t}}}) + \sum_{m=1}^{n-1} \bar{v}_{m+1} h_{n-m}(-2\tilde{\partial}_{\bar{\mathbf{t}}}) + \bar{\partial}_1 h_n(-2\tilde{\partial}_{\bar{\mathbf{t}}}) \right) \Phi &= 0 \end{aligned} \quad (4.23)$$

for $n = 2, 3, \dots$, where $\tilde{\partial}_{\mathbf{t}}$ and $\tilde{\partial}_{\bar{\mathbf{t}}}$ denote the set of operators

$$\tilde{\partial}_{\mathbf{t}} = \left(\partial_1, \frac{\partial_3}{3}, \dots, \frac{\partial_{2n+1}}{2n+1}, \dots \right), \quad \tilde{\partial}_{\bar{\mathbf{t}}} = \left(\bar{\partial}_1, \frac{\bar{\partial}_3}{3}, \dots, \frac{\bar{\partial}_{2n+1}}{2n+1}, \dots \right),$$

and v_n 's and \bar{v}_n 's are defined by generating functions as

$$\begin{aligned} \partial_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) &= \lambda + \sum_{n=1}^{\infty} v_{n+1} \lambda^{-n}, \\ \bar{\partial}_1 \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda) &= \lambda + \sum_{n=1}^{\infty} \bar{v}_{n+1} \lambda^{-n}. \end{aligned}$$

For example, the equations for $n = 2$ in (4.23) take the form

$$\begin{aligned} \left(-\frac{2}{3} \partial_3 + \frac{2}{3} \partial_1^3 + 4(\partial_1^2 \log \tau) \partial_1 \right) \Phi &= 0, \\ \left(-\frac{2}{3} \bar{\partial}_3 + \frac{2}{3} \bar{\partial}_1^3 + 4(\bar{\partial}_1^2 \log \tau) \bar{\partial}_1 \right) \Phi &= 0 \end{aligned}$$

or, equivalently,

$$\begin{aligned} (\partial_3 - \partial_1^3 - 6(\partial_1^2 \log \tau) \partial_1) \Phi &= 0, \\ (\bar{\partial}_3 - \bar{\partial}_1^3 - 6(\bar{\partial}_1^2 \log \tau) \bar{\partial}_1) \Phi &= 0, \end{aligned}$$

which coincide with the lowest equations of (4.14). The equations for $n = 2k$ in (4.23) take such a form as

$$\begin{aligned} \left(-\frac{2}{2k+1}\partial_{2k+1} - C_m(\partial_1, \dots, \partial_{2k-1})\right)\Phi &= 0, \\ \left(-\frac{2}{2k+1}\bar{\partial}_{2k+1} - \bar{C}_m(\bar{\partial}_1, \dots, \bar{\partial}_{2k-1})\right)\Phi &= 0, \end{aligned}$$

where $C_m(\partial_1, \dots, \partial_{2k-1})$ and $\bar{C}_m(\bar{\partial}_1, \dots, \bar{\partial}_{2k-1})$ are differential operators containing $\partial_1, \dots, \partial_{2k-1}$ and $\bar{\partial}_1, \dots, \bar{\partial}_{2k-1}$, respectively, but no 0-th order term. As explained in the case of the KP hierarchy, one can convert these equations to the usual evolutionary form (4.14).

Still missing is the two-dimensional Schrödinger equation (4.15). To derive this equation, we convert (4.7) and (4.8) to yet another form as follows. As regards (4.7), we first shift \mathbf{t} and $\bar{\mathbf{t}}$ as $\mathbf{t} \rightarrow \mathbf{t} - 2[\lambda^{-1}]$ and $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - 2[\mu^{-1}]$, then multiply it by $e^{\xi(\mathbf{t}, \lambda)}\tau(\mathbf{t} - 2[\lambda^{-1}], \bar{\mathbf{t}})/\tau(\mathbf{t}, \bar{\mathbf{t}})$. This yields the equation

$$(\partial_1 + \partial_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu))e^{-2\bar{D}(\mu)}\Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda) = (\partial_1 - \partial_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu))\Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda).$$

Exchanging λ and μ , we eventually obtain the equation

$$\begin{aligned} (\partial_1 + \partial_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda))e^{-2\bar{D}(\lambda)}\Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ = (\partial_1 - \partial_1 \log \bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \lambda))\Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.24)$$

Similarly, shifting \mathbf{t} and $\bar{\mathbf{t}}$ as $\mathbf{t} \rightarrow \mathbf{t} - 2[\lambda^{-1}]$ and $\bar{\mathbf{t}} \rightarrow \bar{\mathbf{t}} - 2[\mu]$ in (4.8) and multiply it by $e^{\xi(\bar{\mathbf{t}}, \mu)}\tau(\mathbf{t}, \bar{\mathbf{t}} - 2[\mu])/\tau(\mathbf{t}, \bar{\mathbf{t}})$, we obtain the equation

$$\begin{aligned} (\bar{\partial}_1 + \bar{\partial}_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda))e^{-2D(\lambda)}\bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) \\ = (\bar{\partial}_1 - \bar{\partial}_1 \log \Psi(\mathbf{t}, \bar{\mathbf{t}}, \lambda))\bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu). \end{aligned} \quad (4.25)$$

Expanded in powers of λ , these equations give rise to an infinite number of linear equations. Among them, the equations from the λ^{-1} -terms give exactly the two-dimensional Schrödinger equation (4.15):

$$\begin{aligned} (\partial_1 \bar{\partial}_1 + 2(\partial_1 \bar{\partial}_1 \log \tau))\Psi(\mathbf{t}, \bar{\mathbf{t}}, \mu) &= 0, \\ (\partial_1 \bar{\partial}_1 + 2(\partial_1 \bar{\partial}_1 \log \tau))\bar{\Psi}(\mathbf{t}, \bar{\mathbf{t}}, \mu) &= 0. \end{aligned}$$

Thus the usual auxiliary linear equations (4.14) and (4.15) can be derived from the differential Fay identities. It should be noted that we have not fully used the generating functional equations (4.19)–(4.22), (4.24) and (4.25). Namely, only even power terms of λ in the first four equations and the λ^{-1} -terms in the last two equations are enough to recover all relevant auxiliary linear equations. What meaning, then, do other terms have? The fact is that the other terms give no new equations. For example, the equations for $n = 3$ in (4.23) can be derived from the the corresponding equation for $n = 2$ by differentiating by t_1 and \bar{t}_1 , respectively.

4.5 Dispersionless Hirota equations and Hamilton-Jacobi equations

The procedure of quasi-classical limit is a straightforward generalization of the case of the one-component BKP hierarchy [51].

As regards the Hirota formalism, we assume the quasi-classical ansatz

$$\tau_h(\mathbf{t}, \bar{\mathbf{t}}) = \exp\left(\hbar^{-2}F(\mathbf{t}, \bar{\mathbf{t}}) + O(\hbar^{-1})\right) \quad (4.26)$$

for the rescaled tau function $\tau_h(\mathbf{t}, \bar{\mathbf{t}}) = \tau(\hbar, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}})$. By this ansatz, we can derive the following dispersionless Hirota equations from the foregoing four differential Fay identities [47]:

$$\begin{aligned} & \left(\lambda + \mu - 2\partial_1(D(\lambda) + D(\mu))F\right)e^{4D(\lambda)D(\mu)F} \\ &= \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - 2\partial_1(D(\lambda) - D(\mu))F\right), \end{aligned} \quad (4.27)$$

$$\begin{aligned} & \left(\lambda + \mu - 2\bar{\partial}_1(\bar{D}(\lambda) + \bar{D}(\mu))F\right)e^{4\bar{D}(\lambda)\bar{D}(\mu)F} \\ &= \frac{\lambda + \mu}{\lambda - \mu} \left(\lambda - \mu - 2\bar{\partial}_1(\bar{D}(\lambda) - \bar{D}(\mu))F\right), \end{aligned} \quad (4.28)$$

$$\left(\lambda - 2\partial_1(D(\lambda) + \bar{D}(\mu))F\right)e^{4D(\lambda)\bar{D}(\mu)F} = \lambda - 2\partial_1(D(\lambda) - \bar{D}(\mu))F, \quad (4.29)$$

$$\left(\mu - 2\bar{\partial}_1(D(\lambda) + \bar{D}(\mu))F\right)e^{4D(\lambda)\bar{D}(\mu)F} = \mu - 2\bar{\partial}_1(\bar{D}(\mu) - D(\lambda))F. \quad (4.30)$$

By defining the S functions $S(z) = S(\mathbf{t}, \bar{\mathbf{t}}, z)$ and $\bar{S}(z) = \bar{S}(\mathbf{t}, \bar{\mathbf{t}}, z)$ as

$$S(z) = \xi(\mathbf{t}, z) - 2D(z)F, \quad \bar{S}(z) = \xi(\bar{\mathbf{t}}, z) - 2\bar{D}(z)F,$$

(4.27) – (4.30) can be rewritten as

$$(\partial_1 S(\lambda) + \partial_1 S(\mu))e^{4D(\lambda)D(\mu)F} = \frac{\lambda + \mu}{\lambda - \mu}(\partial_1 S(\lambda) - \partial_1 S(\mu)), \quad (4.31)$$

$$(\bar{\partial}_1 \bar{S}(\lambda) + \bar{\partial}_1 \bar{S}(\mu))e^{4\bar{D}(\lambda)\bar{D}(\mu)F} = \frac{\lambda + \mu}{\lambda - \mu}(\bar{\partial}_1 \bar{S}(\lambda) - \bar{\partial}_1 \bar{S}(\mu)), \quad (4.32)$$

$$(\partial_1 S(\lambda) + \partial_1 \bar{S}(\mu))e^{4D(\lambda)\bar{D}(\mu)F} = \partial_1 S(\lambda) - \partial_1 \bar{S}(\mu), \quad (4.33)$$

$$(\bar{\partial}_1 \bar{S}(\mu) + \bar{\partial}_1 S(\lambda))e^{4D(\lambda)\bar{D}(\mu)F} = \bar{\partial}_1 \bar{S}(\mu) - \bar{\partial}_1 S(\lambda). \quad (4.34)$$

Actually, (4.31) was first obtained by Bogdanov and Konopelchenko [52] by the $\bar{\partial}$ -dressing method, and rederived, along with the other equations, by Takasaki [47] by the method presented here. Later on, Chen and Tu [53] studied these equations by the kernel method of Carroll and Kodama [12].

Quasi-classical limit is also achieved in the Lax formalism. The ansatz for the wave functions again takes the WKB form

$$\begin{aligned}\Psi_{\hbar}(\mathbf{t}, \bar{\mathbf{t}}, z) &= \exp\left(\hbar^{-1}S(z) + O(\hbar^0)\right), \\ \bar{\Psi}_{\hbar}(\mathbf{t}, \bar{\mathbf{t}}, z) &= \exp\left(\hbar^{-1}\bar{S}(z) + O(\hbar^0)\right)\end{aligned}\tag{4.35}$$

for the rescaled wave functions $\Psi_{\hbar}(\mathbf{t}, \bar{\mathbf{t}}, z) = \Psi(\hbar, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}}, z)$ and $\bar{\Psi}_{\hbar}(\mathbf{t}, \bar{\mathbf{t}}, z) = \bar{\Psi}(\hbar, \hbar^{-1}\mathbf{t}, \hbar^{-1}\bar{\mathbf{t}}, z)$. The foregoing S functions $S = S(z), \bar{S}(z)$ show up here as the phase functions of the WKB approximation, and satisfy the Hamilton-Jacobi equations

$$\begin{aligned}\partial_{2n+1}S &= \mathcal{B}_{2n+1}(\partial_1 S), & \bar{\partial}_{2n+1}S &= \bar{\mathcal{B}}_{2n+1}(\bar{\partial}_1 S), \\ \partial_1 \bar{\partial}_1 S &= u\end{aligned}\tag{4.36}$$

with the classical Hamiltonians

$$\mathcal{B}_{2n+1}(p) = \lim_{\hbar \rightarrow 0} B_{\hbar, 2n+1}(p), \quad \bar{\mathcal{B}}_{2n+1}(\bar{p}) = \lim_{\hbar \rightarrow 0} \bar{B}_{\hbar, 2n+1}(\bar{p})$$

obtained from the (rescaled) operators $B_{\hbar, 2n+1} = B_{\hbar, 2n+1}(\hbar \partial_1)$ and $\bar{B}_{\hbar, 2n+1} = \bar{B}_{\hbar, 2n+1}(\hbar \bar{\partial}_1)$. Hamilton-Jacobi equations of this type were also studied by Konopelchenko and Moro [54] in the context of nonlinear optics.

Though the structure of the dispersionless Hirota equations look considerably different from those of the dispersionless KP and Toda hierarchies, one can anyhow show that they are, in fact, equivalent to the Hamilton-Jacobi equations [47].

5 DKP hierarchy and differential Fay identities

5.1 Bilinear equations for tau function

Our final case study is focused on the DKP hierarchy. This is a subsystem of one of Jimbo and Miwa's integrable hierarchies with with D_{∞} symmetries [55]. The master system proposed by Jimbo and Miwa is formulated in terms of tau functions $\tau(s, \mathbf{t})$ with a discrete variable $s \in \mathbf{Z}$ and a set of continuous variables $\mathbf{t} = (t_1, t_2, \dots)$. These tau functions satisfy an infinite number of Hirota equations (or an equivalent set of bilinear equations of the contour integral type), which can be divided to three subsets, namely, the equations for $\tau(2s, \mathbf{t})$'s, the equations for $\tau(2s+1, \mathbf{t})$'s and the equations of the coupled type inbetween. The Hirota equations of the first and second types are actually identical; it is this system of Hirota equations that we call "the DKP hierarchy". Thus Jimbo and

Miwa's master system contains two copies of the DKP hierarchy as subsystems, and may be called a two-component DKP hierarchy.

Since being a relatively less known member of the big family of integrable hierarchies, the DKP hierarchy has been rediscovered with a different name such as “the coupled KP hierarchy” [56, 57, 58, 59] and “the Pfaff lattice” [60, 61]. Kac and van de Leur [45] called Jimbo and Miwa's master system “the charged BKP hierarchy”, and formulated a multi-component hierarchy in which the BKP, DKP and charged BKP hierarchies and their multi-component analogues are unified.

Let $\tau(s, \mathbf{t})$ denote the tau function of the DKP hierarchy. Note that it amounts to $\tau(2s, \mathbf{t})$ or $\tau(2s+1, \mathbf{t})$ in Jimbo and Miwa's master system. This tau function satisfies the bilinear equation

$$\oint \frac{dz}{2\pi i} z^{2s'-2s-2} e^{\xi(\mathbf{t}'-\mathbf{t}, z)} \tau(s'-1, \mathbf{t}' - [z^{-1}]) \tau(s, \mathbf{t} + [z^{-1}]) \\ + \oint \frac{dz}{2\pi i} z^{2s-2s'-2} e^{\xi(\mathbf{t}-\mathbf{t}', z)} \tau(s', \mathbf{t}' + [z^{-1}]) \tau(s-1, \mathbf{t} - [z^{-1}]) = 0. \quad (5.1)$$

Here and in the following, the notations are mostly the same as used for the KP hierarchy. The integrals on the left hand side are contour integrals along a sufficiently large circle $|z| = R$ (or a formal algebraic operator extracting the coefficient of λ^{-1}). As in the other cases, this bilinear equation, too, can be converted to the Hirota form

$$\sum_{n=0}^{\infty} h_n(-2\mathbf{a}) h_{n+2s'-2s-1}(\tilde{D}\mathbf{t}) e^{\langle \mathbf{a}, D\mathbf{t} \rangle} \tau(s, \mathbf{t}) \cdot \tau(s'-1, \mathbf{t}) \\ + \sum_{n=0}^{\infty} h_n(2\mathbf{a}) h_{n+2s-2s'-1}(-\tilde{D}\mathbf{t}) e^{\langle \mathbf{a}, D\mathbf{t} \rangle} \tau(s-1, \mathbf{t}) \cdot \tau(s', \mathbf{t}) = 0. \quad (5.2)$$

For example, if $s' = s+1$, terms linear in a_n give the special Hirota equations

$$(D_1 D_n - 2h_{n+1}(\tilde{D}\mathbf{t})) \tau(s, \mathbf{t}) \cdot \tau(s, \mathbf{t}) \\ - 2h_{n-3}(-\tilde{D}\mathbf{t}) \tau(s-1, \mathbf{t}) \cdot \tau(s+1, \mathbf{t}) = 0 \quad (5.3)$$

for $n = 3, 4, \dots$

Note that the left hand side of (5.3) coincide with that of the special Hirota equations (2.4) of the KP hierarchy. In this respect, the right hand side may be thought of as a coupling term among the tau functions $\tau(s, \mathbf{t})$ with different indices; the name “coupled KP hierarchy” originates in such an interpretation. This is, however, a rather superficial analogy. Both systems are considerably different. For example, as the name “Pfaff lattice” indicates, its relevant equations and solutions exhibit a Pfaffian structure [56, 57, 58, 60, 61, 62, 63, 64, 65] unlike the determinantal structure of the KP hierarchy.

Let us mention that the BKP hierarchy is also known to have a Pfaffian structure. The difference between the BKP and DKP hierarchies stems from

the type of free fermions behind the tau functions [55, 45]. Namely, whereas the BKP hierarchy is related to neutral fermions, the DKP hierarchy is formulated in terms of charged fermions. The charged fermion system is also used for the KP hierarchy; this explains the superficial similarity between the KP and DKP hierarchies. Actually, Jimbo and Miwa [55] used three types of free fermion systems (charged fermions, neutral fermions and two-component charged fermions) to formulate three integrable hierarchies with D_∞ symmetries. The DKP hierarchy is embedded in their first hierarchy formulated by charged fermions. It is interesting that their second hierarchy, formulated by neutral fermions, contains two copies of the two-component BKP hierarchy.

5.2 Differential Fay identities

Two differential Fay identities are obtained from the bilinear equation (5.1) by the following procedure:

- (i) Differentiate the bilinear equation by t'_1 and specialize s', \mathbf{t}' as $s' = s + 1$ and $\mathbf{t}' = \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]$.
- (ii) Differentiate the bilinear equation by t'_1 and specialize s', \mathbf{t}' as $s' = s$ and $\mathbf{t}' = \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]$.

By (i), the bilinear equation reduces to

$$\oint \frac{dz}{2\pi i} \frac{\lambda\mu}{(z-\lambda)(z-\mu)} \left(z\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \right. \\ \left. + (\partial_1 \tau)(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}]) \right) \tau(s, \mathbf{t} + [z^{-1}]) \\ + \oint \frac{dz}{2\pi i} \frac{(z-\lambda)(z-\mu)}{\lambda\mu z^4} \left(-z\tau(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] + [z^{-1}]) \right. \\ \left. + (\partial_1 \tau)(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}] + [z^{-1}]) \right) \tau(s-1, \mathbf{t} - [z^{-1}]) = 0.$$

Residue calculus and some algebra yield the equation

$$\frac{\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}])\tau(s, \mathbf{t})}{\tau(s, \mathbf{t} + [\lambda^{-1}])\tau(s, \mathbf{t} + [\mu^{-1}])} \\ - \frac{1}{\lambda^2\mu^2} \frac{\tau(s+1, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}])\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t} + [\lambda^{-1}])\tau(s, \mathbf{t} + [\mu^{-1}])} \\ = 1 - \frac{1}{\lambda - \mu} \partial_1 \log \frac{\tau(s, \mathbf{t} + [\lambda^{-1}])}{\tau(s, \mathbf{t} + [\mu^{-1}])}. \quad (5.4)$$

In the same way, (ii) leads to the equation

$$\begin{aligned} & \frac{\lambda^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t}) \tau(s-1, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}])}{\tau(s, \mathbf{t} + [\lambda^{-1}]) \tau(s-1, \mathbf{t} - [\mu^{-1}])} \\ & - \frac{\mu^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}]) \tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t} + [\lambda^{-1}]) \tau(s-1, \mathbf{t} - [\mu^{-1}])} \\ & = \lambda + \mu - \partial_1 \log \frac{\tau(s, \mathbf{t} + [\lambda^{-1}])}{\tau(s-1, \mathbf{t} - [\mu^{-1}])}, \end{aligned}$$

but from the point of view of symmetry, it will be better to shift \mathbf{t} as $\mathbf{t} \rightarrow \mathbf{t} + [\mu^{-1}]$. This gives the equation

$$\begin{aligned} & \frac{\lambda^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t} + [\mu^{-1}]) \tau(s-1, \mathbf{t} + [\lambda^{-1}])}{\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]) \tau(s-1, \mathbf{t})} \\ & - \frac{\mu^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t} + [\lambda^{-1}]) \tau(s-1, \mathbf{t} + [\mu^{-1}])}{\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]) \tau(s-1, \mathbf{t})} \\ & = \lambda + \mu - \partial_1 \log \frac{\tau(s, \mathbf{t} + [\lambda^{-1}] + [\mu^{-1}])}{\tau(s-1, \mathbf{t})}. \quad (5.5) \end{aligned}$$

(5.4) was first derived by Adler, Horozov and van Moerbeke in their study of the Pfaff lattice [60]. As they pointed out, this equation resembles the differential Fay identity (2.7) of the KP hierarchy; this is another aspect of the aforementioned similarity with the KP hierarchy. The only difference is the presence of the second term on the left hand side. This is, however, an essential difference.

(5.5) has no analogue in the KP hierarchy, etc. A remarkable characteristic of this equation and (5.4) is that there are two terms, rather than just one, that contain the double shift $\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]$ of \mathbf{t} . This apparently small difference eventually affects every aspect of this system and its dispersionless analogue.

Lastly, using the operator $D(z)$, we can rewrite these differential Fay identities as follows:

$$\begin{aligned} & \exp\left((e^{D(\lambda)} - 1)(e^{D(\mu)} - 1) \log \tau\right) \\ & - \frac{1}{\lambda^2 \mu^2} \exp\left((e^{D(\lambda)+\partial_s} - 1)(e^{D(\mu)+\partial_s} - 1) e^{-\partial_s} \log \tau\right) \\ & = 1 - \frac{\partial_1(e^{D(\lambda)} - e^{D(\mu)}) \log \tau}{\lambda - \mu}, \quad (5.6) \end{aligned}$$

$$\begin{aligned} & \frac{\lambda^2}{\lambda - \mu} \exp\left(-(e^{D(\lambda)} - 1)(e^{D(\mu)} - e^{-\partial_s}) \log \tau\right) \\ & - \frac{\mu^2}{\lambda - \mu} \exp\left(-(e^{D(\lambda)} - e^{-\partial_s})(e^{D(\mu)} - 1) \log \tau\right) \\ & = \lambda + \mu - \partial_1(e^{D(\lambda)+D(\mu)} - e^{-\partial_s}) \log \tau. \quad (5.7) \end{aligned}$$

5.3 Auxiliary linear equations

The rather complicated structure of the differential Fay identities is closely related to the fact that an underlying auxiliary linear problem is vector-valued. This auxiliary linear problem is substantially the same as the one first discovered by Kakei [58] in an inverse scattering formalism, and just a special case of the work of Kac and van de Leur [45] on multi-component DKP/BKP hierarchies.

We introduce the wave functions

$$\begin{aligned}\Psi_1(s, \mathbf{t}, z) &= \frac{\tau(s, \mathbf{t} - [z^{-1}])}{\tau(s, \mathbf{t})} z^{2s} e^{\xi(\mathbf{t}, z)}, \\ \Psi_2(s, \mathbf{t}, z) &= \frac{\tau(s-1, \mathbf{t} - [z^{-1}])}{\tau(s, \mathbf{t})} z^{2s-2} e^{\xi(\mathbf{t}, z)}\end{aligned}$$

and the duals

$$\begin{aligned}\Psi_1^*(s, \mathbf{t}, z) &= \frac{\tau(s+1, \mathbf{t} + [z^{-1}])}{\tau(s, \mathbf{t})} z^{-2s-2} e^{-\xi(\mathbf{t}, z)}, \\ \Psi_2^*(s, \mathbf{t}, z) &= \frac{\tau(s-1, \mathbf{t} + [z^{-1}])}{\tau(s, \mathbf{t})} z^{-2s} e^{-\xi(\mathbf{t}, z)}.\end{aligned}$$

By suitably shifting s' and s , the bilinear equation (5.1) for the tau function yields the bilinear equations

$$\begin{aligned}\oint \frac{dz}{2\pi i} \Psi_1(s', \mathbf{t}', z) \Psi_1^*(s, \mathbf{t}, z) + \oint \frac{dz}{2\pi i} \Psi_1^*(s', \mathbf{t}, z) \Psi_1(s, \mathbf{t}, z) &= 0, \\ \oint \frac{dz}{2\pi i} \Psi_1(s', \mathbf{t}', z) \Psi_2^*(s, \mathbf{t}, z) + \oint \frac{dz}{2\pi i} \Psi_1^*(s', \mathbf{t}', z) \Psi_2(s, \mathbf{t}, z) &= 0, \\ \oint \frac{dz}{2\pi i} \Psi_2(s', \mathbf{t}', z) \Psi_1^*(s, \mathbf{t}, z) + \oint \frac{dz}{2\pi i} \Psi_2^*(s', \mathbf{t}', z) \Psi_1(s, \mathbf{t}, z) &= 0, \\ \oint \frac{dz}{2\pi i} \Psi_2(s', \mathbf{t}', z) \Psi_2^*(s, \mathbf{t}, z) + \oint \frac{dz}{2\pi i} \Psi_2^*(s', \mathbf{t}', z) \Psi_2(s, \mathbf{t}, z) &= 0\end{aligned}$$

for these wave functions. Moreover, these four equations can be cast into the matrix form

$$\oint \frac{dz}{2\pi i} \begin{pmatrix} \Psi_1(s', \mathbf{t}', z) & \Psi_1^*(s', \mathbf{t}', z) \\ \Psi_2(s', \mathbf{t}', z) & \Psi_2^*(s', \mathbf{t}', z) \end{pmatrix} \begin{pmatrix} \Psi_1^*(s, \mathbf{t}, z) & \Psi_2^*(s, \mathbf{t}, z) \\ \Psi_1(s, \mathbf{t}, z) & \Psi_2(s, \mathbf{t}, z) \end{pmatrix} = 0. \quad (5.8)$$

This matrix bilinear equation and its building blocks are reminiscent of those of the charged two-component KP hierarchy [30]. There is, however, an unnegligible difference. In the case of the two-component KP hierarchy with charge $(2s, -2s)$, the $(2, 1)$ and $(1, 2)$ elements of the matrix wave functions should have z^{2s-1} and z^{-2s-1} rather than z^{2s-2} and z^{-2s-2} in the foregoing definition of the wave functions. Therefore if one interprets this system as a reduction of the two-component KP hierarchy, one should remember that this is a very special reduction. Actually, Adler, Horozov and van Moerbeke [60] obtained the Pfaff lattice in a “deep stratum” of the phase space of the Toda hierarchy.

As Kac and van de Leur did in a more general setting [45], one can derive auxiliary linear equations of the form

$$\partial_n \begin{pmatrix} \Psi_1(s, \mathbf{t}, z) & \Psi_1^*(s, \mathbf{t}, z) \\ \Psi_2(s, \mathbf{t}, z) & \Psi_2^*(s, \mathbf{t}, z) \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} \Psi_1(s, \mathbf{t}, z) & \Psi_1^*(s, \mathbf{t}, z) \\ \Psi_2(s, \mathbf{t}, z) & \Psi_2^*(s, \mathbf{t}, z) \end{pmatrix}, \quad (5.9)$$

where $A_n = A_n(s, \mathbf{t}, \partial_1)$, $B_n = B_n(s, \mathbf{t}, \partial_1)$, $C_n = C_n(s, \mathbf{t}, \partial_1)$, $D_n = D_n(s, \mathbf{t}, \partial_1)$ are differential operator with respect to t_1 . One can specify the structure of these operators in more detail by introducing dressing operators for the wave functions. In particular, as Kakei observed [58], A_n, B_n, C_n, D_n turn out to be operators of the form

$$\begin{aligned} A_n &= \partial_1^n + O(\partial_1^{n-2}), & B_n &= O(\partial_1^{n-2}), \\ C_n &= O(\partial_1^{n-2}), & D_n &= -(-\partial_1)^n + O(\partial_1^{n-2}) \end{aligned} \quad (5.10)$$

and satisfy the algebraic relations

$$\begin{aligned} A_n^* &= -D_n, & D_n^* &= -A_n, \\ B_n^* &= B_n, & C_n^* &= C_n, \end{aligned} \quad (5.11)$$

where $A_n^*, B_n^*, C_n^*, D_n^*$ denote the formal adjoint of A_n, B_n, C_n, D_n .

5.4 Differential Fay identities and auxiliary linear equations

We now translate the differential Fay identities (5.4) and (5.5) to the language of the wave functions. As it turns out, each of the differential Fay identities can be converted to two different equations (hence altogether four equations) for the wave functions. These equations give a generating functional expression of the auxiliary linear equations presented above.

Let us first consider (5.4). Shifting \mathbf{t} as $\mathbf{t} \rightarrow \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}]$, we have the equation

$$\begin{aligned} & \frac{\tau(s, \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}])\tau(s, \mathbf{t})}{\tau(s, \mathbf{t} - [\lambda^{-1}])\tau(s, \mathbf{t} - [\mu^{-1}])} \\ & - \frac{1}{\lambda^2 \mu^2} \frac{\tau(s+1, \mathbf{t})\tau(s-1, \mathbf{t} - [\lambda^{-1}] - [\mu^{-1}])}{\tau(s, \mathbf{t} - [\lambda^{-1}])\tau(s, \mathbf{t} - [\mu^{-1}])} \\ & = 1 + \frac{1}{\lambda - \mu} \partial_1 \log \frac{\tau(s, \mathbf{t} - [\lambda^{-1}])}{\tau(s, \mathbf{t} - [\mu^{-1}])}, \end{aligned}$$

and multiplying this equation by $(\lambda - \mu)\mu^{2s}e^{\xi(\mathbf{t}, \mu)}\tau(s, \mathbf{t} - [\mu^{-1}])/\tau(s, \mathbf{t})$, we obtain the equation

$$\begin{aligned} & \lambda e^{-D(\lambda)}\Psi_1(s, \mathbf{t}, \mu) - \lambda^{-1} \frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{-D(\lambda)}\Psi_2(s, \mathbf{t}, \mu) \\ & = (\partial_1 \log \Psi_1(s, \mathbf{t}, \lambda) - \partial_1)\Psi_1(s, \mathbf{t}, \mu). \end{aligned} \quad (5.12)$$

Similarly, multiplying both hand side of (5.4) (without shifting \mathbf{t}) by $(\lambda - \mu)\mu^{-2s}e^{-\xi(\mathbf{t},\mu)}\tau(s, \mathbf{t} + [\mu^{-1}])/\tau(s, \mathbf{t})$ yields the equation

$$\begin{aligned} \lambda e^{D(\lambda)}\Psi_2^*(s, \mathbf{t}, \mu) - \lambda^{-1} \frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{D(\lambda)}\Psi_1^*(s, \mathbf{t}, \mu) \\ = (\partial_1 - \partial_1 \log \Psi_2^*(s, \mathbf{t}, \lambda))\Psi_2^*(s, \mathbf{t}, \mu). \end{aligned} \quad (5.13)$$

Note that, unlike the previous cases, there are two terms containing $e^{\pm D(\lambda)}$. They stem from the two terms in (5.4) containing the double shift $\mathbf{t} + [\lambda^{-1}] + [\mu^{-1}]$ of \mathbf{t} .

We now consider (5.5) or, rather its original form

$$\begin{aligned} \frac{\lambda^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t})\tau(s-1, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}])}{\tau(s, \mathbf{t} + [\lambda^{-1}])\tau(s-1, \mathbf{t} - [\mu^{-1}])} \\ - \frac{\mu^2}{\lambda - \mu} \frac{\tau(s, \mathbf{t} + [\lambda^{-1}] - [\mu^{-1}])\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t} + [\lambda^{-1}])\tau(s-1, \mathbf{t} - [\mu^{-1}])} \\ = \lambda + \mu - \partial_1 \log \frac{\tau(s, \mathbf{t} + [\lambda^{-1}])}{\tau(s-1, \mathbf{t} - [\mu^{-1}])}. \end{aligned}$$

Multiplying this equation by $\lambda^{-2s}e^{-\xi(\mathbf{t},\lambda)}\tau(s, \mathbf{t} + [\lambda^{-1}])/\tau(s-1, \mathbf{t})$ yields the equation

$$\begin{aligned} \mu^{-1} \frac{\tau(s, \mathbf{t})}{\tau(s-1, \mathbf{t})} e^{-D(\mu)}\Psi_2^*(s, \mathbf{t}, \lambda) - \mu e^{-D(\mu)}\Psi_1^*(s-1, \mathbf{t}, \lambda) \\ = (\partial_1 - \partial_1 \log \Psi_1(s-1, \mathbf{t}, \mu))\Psi_1^*(s-1, \mathbf{t}, \lambda). \end{aligned}$$

Exchanging λ and μ and shifting s as $s \rightarrow s+1$, we eventually obtain the equation

$$\begin{aligned} \lambda e^{-D(\lambda)}\Psi_1^*(s, \mathbf{t}, \mu) - \lambda^{-1} \frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{-D(\lambda)}\Psi_2^*(s, \mathbf{t}, \mu) \\ = (\partial_1 \log \Psi_1(s, \mathbf{t}, \lambda) - \partial_1)\Psi_1^*(s, \mathbf{t}, \mu), \end{aligned} \quad (5.14)$$

which has the same structure as (5.12). Similarly, multiplying the same equation as above by $\mu^{2s-2}e^{\xi(\mathbf{t},\mu)}\tau(s-1, \mathbf{t} - [\mu^{-1}])/\tau(s, \mathbf{t})$, we obtain the equation

$$\begin{aligned} \lambda e^{D(\lambda)}\Psi_2(s, \mathbf{t}, \mu) - \lambda^{-1} \frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{D(\lambda)}\Psi_1(s, \mathbf{t}, \mu) \\ = (\partial_1 - \partial_1 \log \Psi_2^*(s, \mathbf{t}, \lambda))\Psi_2(s, \mathbf{t}, \mu), \end{aligned} \quad (5.15)$$

which can be compared with (5.13).

These results (5.12) – (5.15) show that the vector-valued wave functions

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Psi_1(s, \mathbf{t}, \mu) \\ \Psi_2(s, \mathbf{t}, \mu) \end{pmatrix}, \quad \begin{pmatrix} \Psi_1^*(s, \mathbf{t}, \mu) \\ \Psi_2^*(s, \mathbf{t}, \mu) \end{pmatrix}$$

satisfy the same linear equations

$$\begin{aligned}\lambda e^{-D(\lambda)}\Phi_1 - \lambda^{-1} \frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{-D(\lambda)}\Phi_2 &= (\partial_1 \log \Psi_1(s, \mathbf{t}, \lambda) - \partial_1)\Phi_1, \\ \lambda e^{D(\lambda)}\Phi_2 - \lambda^{-1} \frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})} e^{D(\lambda)}\Phi_1 &= (\partial_1 - \partial_1 \log \Psi_2^*(s, \mathbf{t}, \lambda))\Phi_2.\end{aligned}$$

Expanded in powers of λ , these equations give rise to an infinite number of linear equations of the form

$$\begin{aligned}h_n(-\tilde{\partial}_{\mathbf{t}})\Phi_1 - \frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})} h_{n-2}(-\tilde{\partial}_{\mathbf{t}})\Phi_2 &= v_n \Phi_1, \\ h_n(\tilde{\partial}_{\mathbf{t}})\Phi_2 - \frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})} h_{n-2}(\tilde{\partial}_{\mathbf{t}})\Phi_1 &= u_n \Phi_2\end{aligned}\tag{5.16}$$

for $n = 2, 3, \dots$, where v_n and u_n are defined by

$$v_n = \partial_1 h_{n-2}(-\tilde{\partial}_{\mathbf{t}}) \log \tau(s, \mathbf{t}), \quad u_n = \partial_1 h_{n-2}(\tilde{\partial}_{\mathbf{t}}) \log \tau(s, \mathbf{t}).$$

The lowest equations (for $n = 2$) read

$$\begin{aligned}\frac{1}{2}(\partial_1^2 - \partial_2)\Phi_1 - \frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})}\Phi_2 &= -(\partial_1^2 \log \tau(s, \mathbf{t}))\Phi_1, \\ \frac{1}{2}(\partial_1^2 + \partial_2)\Phi_2 - \frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})}\Phi_1 &= -(\partial_1^2 \log \tau(s, \mathbf{t}))\Phi_2,\end{aligned}$$

which one can rewrite as

$$\partial_2 \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \partial_1^2 + 2\partial_1^2 \log \tau(s, \mathbf{t}) & -2\frac{\tau(s+1, \mathbf{t})}{\tau(s, \mathbf{t})} \\ 2\frac{\tau(s-1, \mathbf{t})}{\tau(s, \mathbf{t})} & \partial_1^2 - 2\partial_1^2 \log \tau(s, \mathbf{t}) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

This is exactly the lowest equation of the auxiliary linear problem (5.9). Higher equations can be derived recursively by the same procedure as in the case of the KP hierarchy etc.

5.5 Quasi-classical limit — encounter with difficulty

Quasi-classical limit can be achieved by assuming the ansatz

$$\tau_{\hbar}(s, \mathbf{t}) = \exp\left(\hbar^{-2}F(s, \mathbf{t}) + O(\hbar^{-1})\right)\tag{5.17}$$

for the rescaled tau function $\tau_{\hbar}(s, \mathbf{t}) = \tau(\hbar, \hbar^{-1}s, \hbar^{-1}\mathbf{t})$. The differential Fay identities, written in the form of (5.6) and (5.7), yield the dispersionless Hirota equations

$$e^{D(\lambda)D(\mu)F} - \lambda^{-2}\mu^{-2}e^{(D(\lambda)+\partial_s)(D(\mu)+\partial_s)F} = 1 - \frac{\partial_1(D(\lambda) - D(\mu))F}{\lambda - \mu},\tag{5.18}$$

$$\begin{aligned} \frac{\lambda^2}{\lambda - \mu} e^{-D(\lambda)(D(\mu) + \partial_s)F} - \frac{\mu^2}{\lambda - \mu} e^{-(D(\lambda) + \partial_s)D(\mu)F} \\ = \lambda + \mu - \partial_1(D(\lambda) + D(\mu) + \partial_s)F \end{aligned} \quad (5.19)$$

for the F function. The first equation obviously resembles its counterpart (2.23) for the KP hierarchy; the only difference is the presence of the second term on the left hand side. On the other hand, the second equation seems to have no analogue in the previous cases.

Let us examine these equations in more detail. It is convenient (and natural) to introduce the S function

$$S(z) = \xi(\mathbf{t}, z) + 2s \log z - D(z)F$$

and rewrite the two equations as

$$e^{D(\lambda)D(\mu)F} (1 - e^{\partial_s^2 F - \partial_s S(\lambda) - \partial_s S(\mu)}) = \frac{\partial_1 S(\lambda) - \partial_1 S(\mu)}{\lambda - \mu}, \quad (5.20)$$

$$\frac{e^{-D(\lambda)D(\mu)F}}{\lambda - \mu} (e^{\partial_s S(\lambda)} - e^{\partial_s S(\mu)}) = \partial_1 S(\lambda) + \partial_1 S(\mu) - \partial_s \partial_1 F. \quad (5.21)$$

One can eliminate $e^{D(\lambda)D(\mu)F}$ by multiplying both hand sides of these equations. This yields the relation

$$\begin{aligned} (1 - e^{\partial_s^2 F - \partial_s S(\lambda) - \partial_s S(\mu)}) (e^{\partial_s S(\lambda)} - e^{\partial_s S(\mu)}) \\ = (\partial_1 S(\lambda) - \partial_1 S(\mu)) (\partial_1 S(\lambda) + \partial_1 S(\mu) - \partial_s \partial_1 F), \end{aligned}$$

which one can rewrite as

$$\begin{aligned} (\partial_1 S(\lambda))^2 - (\partial_1 \partial_s F)(\partial_1 S(\lambda)) - e^{\partial_s S(\lambda)} - e^{\partial_s^2 F - \partial_s S(\lambda)} \\ = (\partial_1 S(\mu))^2 - (\partial_1 \partial_s F)(\partial_1 S(\mu)) - e^{\partial_s S(\mu)} - e^{\partial_s^2 F - \partial_s S(\mu)}. \end{aligned}$$

Since λ and μ are separated to the left hand side and the right hand side, both hand sides of this equation are actually independent of λ and μ . One can determine their value by letting $\lambda, \mu \rightarrow \infty$. We thus obtain the equation

$$\begin{aligned} (\partial_1 S(\mu))^2 - (\partial_1 \partial_s F)(\partial_1 S(\mu)) - e^{\partial_s S(\mu)} - e^{\partial_s^2 F - \partial_s S(\mu)} \\ = -2\partial_1^2 F + \frac{1}{2}\partial_2 \partial_s F - \frac{1}{2}(\partial_1 \partial_s F)^2 \end{aligned} \quad (5.22)$$

as a consequence of (5.20) and (5.21). Actually, this procedure can be reversed. Namely, assuming that (5.20) holds, one can easily recover (5.21) from (5.22). Thus under (5.20), (5.21) and (5.22) are equivalent.

(5.22) may be thought of as a quadratic constraint to (5.20). This constraint can be solved for either $\partial_1 S(\mu)$ or $e^{\partial_s S(\mu)}$, which becomes a non-polynomial function of the other. For example, solving it for $e^{\partial_s S(\mu)}$ gives

$$e^{\partial_s S(\mu)} = \frac{1}{2} \left(p(\mu)^2 - (\partial_1 \partial_s F)p(\mu) - f + \sqrt{D} \right), \quad (5.23)$$

where

$$p(\mu) = \partial_1 S(\mu), \quad D = \left(p(\mu)^2 - (\partial_1 \partial_s F) p(\mu) - f \right)^2 - 4e^{\partial_s^2 F},$$

and f denotes the right hand side of (5.22). This enables one to eliminate $e^{\partial_s S(\lambda)}$ and $e^{\partial_s S(\mu)}$ from (5.20). The outcome is an equation for $D(\lambda)D(\mu)F$, $\partial_1 S(\lambda)$ and $\partial_1 S(\mu)$. One can rewrite it in a form that amounts to (2.31). Thus $S(\mu)$ anyhow turns out to satisfy a set of Hamilton-Jacobi equations of the form

$$\partial_n S(\mu) = H_n(\partial_1 S(\mu)). \quad (5.24)$$

A serious problem shows up here. Namely, unlike the cases of the KP, Toda and BKP hierarchies, the right hand side of these Hamilton-Jacobi equations is not a polynomial function of $\partial_1 S(\mu)$ but a fairly complicated (though algebraic) irrational function thereof. Roughly speaking, $H_n(p)$ is a combination of polynomials and the square root of a quartic polynomial, namely,

$$H_n(p) = P_n(p) + Q_n(p)\sqrt{D}, \quad (5.25)$$

where $P_n(p)$ and $Q_n(p)$ are polynomials in p , and D is given by

$$D = (p^2 - (\partial_1 \partial_s F)p - f)^2 - 4e^{\partial_s^2 F}.$$

This new phenomena, hinting at the relevance of an elliptic curve, is closely related to the fact that the auxiliary linear problem (5.9) is essentially vector-valued. Because of this complicated structure of the Hamiltonians $H_n(p)$, the question of integrability of the underlying dispersionless system has not been resolved.

These results could have been derived by the quasi-classical limit of the auxiliary linear equations. Such an approach, however, is technically more subtle, because the auxiliary linear problem has matrix coefficients. The foregoing approach based on the differential Fay identities is obviously simpler and more reliable. This is a demonstration of the advantage of using differential Fay identities.

6 Concluding remarks

One of the most remarkable results of our case studies is that differential (or difference) Fay identities are the auxiliary linear problem in disguise. This interpretation clarifies the meaning of the mysterious linear equations (2.19) of Sato and Sato [29] and its generalization to other integrable hierarchies. The status of differential Fay identities is thus parallel to that of dispersionless Hirota equations. We can use them as a new foundation of dispersive integrable hierarchies. In this new framework, we might be able to go beyond dispersionless (or quasi-classical) limit to study higher orders of \hbar -expansion.

Though the case studies in this paper are limited to the KP, Toda, BKP and DKP hierarchies, these results can be (and have been) generalized to some other

cases. A particularly successful case [66] is the charged multi-component KP hierarchy and its dispersionless analogue (which is identified to be the genus-zero universal Whitham hierarchy [10]). In a sense, this is a multidimensional generalization of the Toda hierarchy as well. The results on the KP and Toda hierarchies can be fully generalized to this case.

Among the cases already studied, the DKP hierarchy and its dispersionless analogue are still posing many open problems. The technical complexity in this case stems from the fact that the auxiliary linear problem is essentially vector-valued. In contrast, the charged multi-component KP hierarchy has actually a scalar-valued auxiliary linear problem, by which one can formulate its dispersionless analogue as the genus-zero universal Whitham hierarchy. Our preliminary consideration in this paper indicates that the dispersionless analogue of the DKP hierarchy might be related to the universal Whitham hierarchy of genus one [10] or fall into one of examples of “quasi-classical deformations of algebraic curves” proposed by Kodama, Konopelchenko and Marninez Alonso [67]. The Toda versions of the DKP hierarchy [68, 69] will inherit a similar problem. These issues deserve to be further studied.

To conclude this paper, let us mention a geometric interpretation of dispersionless Hirota equations. This interpretation is inspired by the recent work of Krichever, Marshakov and Zabrodin [70] (see also the work of Carroll and Kodama [12]). As the recent work of Eynard and Orantin [71] on random matrices indicates, such a geometric point of view will be indispensable when one pursues higher orders of \hbar -expansion of $\log \tau_{\hbar}$. For simplicity, we now consider the case of the dispersionless KP hierarchy, but it is rather straightforward to generalize this interpretation to the dispersionless Toda hierarchy and the universal Whitham hierarchy of genus zero. A clue is to take the limit as $\mu \rightarrow \lambda$ in the dispersionless Hirota equation

$$e^{D(\lambda)D(\mu)F} = \frac{p(\lambda) - p(\mu)}{\lambda - \mu}.$$

This yields the equation

$$e^{D(\lambda)^2 F} = p'(\lambda),$$

by which one can rewrite the dispersionless Hirota equation as

$$\exp\left(-\frac{1}{2}(D(\lambda) - D(\mu))^2 F\right) = \frac{p(\lambda) - p(\mu)}{\sqrt{p'(\lambda)}\sqrt{p'(\mu)}(\lambda - \mu)}. \quad (6.1)$$

Remarkably, the quantity

$$E(\lambda, \mu) = \frac{p(\lambda) - p(\mu)}{\sqrt{p'(\lambda)}\sqrt{p'(\mu)}}$$

showing up on the right hand side is nothing but a local expression of the prime form [32, 72] on the Riemann sphere in a neighborhood of $z = \infty$ with local

coordinate $p(z)^{-1}$. We can thereby rewrite (6.1) as

$$E(\lambda, \mu) = (\lambda - \mu) \exp\left(-\frac{1}{2}(D(\lambda) - D(\mu))^2 F\right). \quad (6.2)$$

This coincides with a formula presented by Krichever et al. [70] for the case of nonzero genera. Moreover, taking the logarithmic second derivative of both hand sides with respect to λ and μ , we obtain an expression of the Bergmann kernel $B(\lambda, \mu) = \partial_\lambda \partial_\mu \log E(\lambda, \mu)$ [32, 72]:

$$B(\lambda, \mu) = \frac{1}{(\lambda - \mu)^2} + D'(\lambda)D'(\mu)F. \quad (6.3)$$

Remarkably, this formula resembles a well known result on the two-point loop correlation function (which corresponds to $D'(\lambda)D'(\mu)F$ in this formula) of the Hermitian random matrix model in the large- N limit. This seems to indicate, as Eynard and Orantin [71] observed in the case of random matrices, that the Bergmann kernel is one of fundamental building blocks of \hbar -expansion of $\log \tau_\hbar$.

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